Integrable Lattices: Random Matrices and Random Permutations*

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Abstract

These lectures present a survey of recent developments in the area of random matrices (finite and infinite) and random permutations. These probabilistic problems suggest matrix integrals (or Fredholm determinants), which arise very naturally as integrals over the tangent space to symmetric spaces, as integrals over groups and finally as integrals over symmetric spaces. An important part of these lectures is devoted to showing that these matrix integrals, upon apropriately adding time-parameters, are natural tau-functions for integrable lattices, like the Toda, Pfaff and Toeplitz lattices, but also for integrable PDE's, like the KdV equation. These matrix integrals or Fredholm determinants also satisfy Virasoro constraints, which combined with the integrable equations lead to (partial) differential equations for the original probabilities.

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0 introduction

The purpose of these lectures is to give a survey of recent interactions between statistical questions and integrable theory. Two types of questions will be tackled here:

(i) Consider a random ensemble of matrices, with certain symmetry conditions to guarantee the reality of the spectrum and subjected to a given statistics. What is the probability that all its eigenvalues belong to a given subset E ? What happens, when the size of the matrices gets very large ? The probabilities here are functions of the boundary points c_i of E.

(ii) What is the statistics of the length of the largest increasing sequence in a random permutation, assuming each permutation is equally probable ? Here, one considers generating functions (over the size of the permutations) for the probability distributions, depending on the variable x.

The main emphasis of these lectures is to show that integrable theory serves as a useful tool for finding equations satisfied by these functions of x, and conversely the probabilities point the way to new integrable systems.

These questions are all related to integrals over spaces of matrices. Such spaces can be classical Lie groups or algebras, symmetric spaces or their tangent spaces. In infinite-dimensional situations, the "∞-fold" integrals get replaced by Fredholm determinants.

During the last decade, astonishing discoveries have been made in a variety of directions. A first striking feature is that these probabilities are all related to Painlevé equations or interesting generalizations. In this way, new and unusual distributions have entered the statistical world.

Another feature is that each of these problems is related to some integrable hierarchy. Indeed, by inserting an infinite set of time variables $t_1, t_2, t_3, \ldots$ in the integrals

or Fredholm determinants - e.g., by introducing appropriate exponentials $e^{\sum_1^\infty t_i y^i}$ in the integral - this probability, as a function of $t_1, t_2, t_3, \ldots$, satisfies an integrable hierarchy. Korteweg-de Vries, KP, Toda lattice equations are only a few examples of such integrable equations.

Typically integrable systems can be viewed as isospectral deformations of differential or difference operators $\mathcal{L}$. Perhaps, one of the most startling discoveries of integrable theory is that $\mathcal{L}$ can be expressed in terms of a single"τ-function" $\tau(t_1, t_2, \ldots)$ (or vector of τ-functions), which satisfy an infinite set of non-linear equations, encapsulated in a single "*bilinear identity*". The t_i account for the commuting flows of this integrable hierarchy. In this way, many interesting classical functions live under the same hat: characters of representations, Θ-functions of algebraic geometry, hypergeometric functions, certain integrals over classical Lie algebras or groups, Fredholm determinants, arising in statistical mechanics, in scattering and random matrix theory! They are all special instances of "*τ-functions*".

The point is that the probabilities or generating functions above, as functions of $t_1, t_2, \ldots$ (after some minor renormalization) are precisely such τ-functions for the corresponding integrable hierarchy and thus automatically satisfy a large set of equations.

These probabilities are very special τ-functions: they happen to be a solution of yet another hierarchy of (linear) equations in the variables t_i and the boundary points c_i, namely $\mathbb{J}_k^{(2)} \tau(t;c) = 0$, where the $\mathbb{J}_k^{(2)}$ form -roughly speaking- a Virasoro-like algebra:

$$\left[\, \mathbb{J}_k^{(2)},\ \mathbb{J}_\ell^{(2)} \right] = (k-\ell)\ \mathbb{J}_{k+\ell}^{(2)} + \ldots$$

The point is that each integrable hierarchy has a natural "*vertex operator*", which automatically leads to a natural Virasoro algebra. Then, eliminating the partial derivatives in t from the two hierarchy of equations, the integrable and the Virasoro hierarchies, and finally setting $t=0$, lead to PDE's or ODE's satisfied by the probabilities.

In the table below, we give an overview of the different problems, discussed in this lecture, the relevant integrals in the second column and the different hierarchies satisfied by the integrals. To fix notation, $\mathcal{H}_\ell$, $\mathcal{S}_\ell$, $\mathcal{T}_\ell$ refer to the Hermitian, symmetric and symplectic ensembles, populated respectively by $\ell \times \ell$ Hermitian matrices , symmetric matrices and self-dual Hermitian matrices, with quaternionic entries. $\mathcal{H}_\ell(E)$, $\mathcal{S}_\ell(E)$, $\mathcal{T}_\ell(E)$ are the corresponding set of matrices, with all spectral points belonging to E. $U(\ell)$ and $O(\ell)$ are the unitary and orthogonal groups respectively. In the table below, $V_t(z) := V_0(z) + \sum t_i z^i$, where $V_0(z)$ stands for the unperturbed problem; in the last integral $\tilde{V}_t(z)$ is a more complicated function of $t_1, t_2, \ldots$ and z, to be specified later.

Probability problem	underlying t-perturbed integral, τ-function of $\longrightarrow$	corresponding integrable hierarchies
$P(M \in \mathcal{H}_n(E))$	$\int_{\mathcal{H}_n(E)} e^{Tr(-V(M)+\sum_1^\infty t_i M^i)} dM$	Toda lattice KP hierarchy
$P(M \in \mathcal{S}_n(E))$	$\int_{\mathcal{S}_n(E)} e^{Tr(-V(M)+\sum_1^\infty t_i M^i)} dM$	Pfaff lattice Pfaff-KP hierarchy
$P(M \in \mathcal{T}_n(E))$	$\int_{\mathcal{T}_n(E)} e^{Tr(-V(M)+\sum_1^\infty t_i M^i)} dM$	Pfaff lattice Pfaff-KP hierarchy
$P((M_1, M_2) \in \mathcal{H}_n(E_1) \times \mathcal{H}_n(E_2))$	$\int_{\mathcal{H}_n^2(E)} dM_1 dM_2$ $e^{-\operatorname{Tr}(V_t(M_1)-V_s(M_2)-cM_1M_2)}$	2d-Toda lattice KP-hierarchy
$P(M \in \mathcal{H}_\infty(E))$	$\det\left(I - K_t(y,z) I_{E^c}(z)\right)$ (Fredholm determinant)	KdV equation
longest increasing sequence in random permutations	$\int_{U(\ell)} e^{Tr \sum_1^\infty (t_i M^i - s_i \bar{M}^i)} dM$	Toeplitz lattice 2d-Toda lattice
longest increasing sequence in random involutions	$\int_{O(\ell)} e^{Tr(xM+\tilde{V}_t(M))} dM$	Toda lattice KP-hierarchy

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1 Matrix integrals, random matrices and permutations

1.1 Tangent space to symmetric spaces and associated random matrix ensembles

Random matrices provided a model for excitation spectra of heavy nuclei at high excitations (Wigner [76], Dyson [27] and Mehta [49]), based on the nuclear experimental data by Porter and Rosenzweig [56]; they observed that the occurrence of two levels, close to each other, is a rare event (level repulsion), showing that the spacing is not Poissonian, as one might expect from a naive point of view.

Random matrix ideas play an increasingly prominent role in mathematics: not only have they come up in the spacings of the zeroes of the Riemann zeta function, but their relevance has been observed in the chaotic Sinai billiard and, more generally, in chaotic geodesic flows. Chaos seems to lead to the "spectral rigidity", typical of the spectral distributions of random matrices, whereas the spectrum of an integrable system is random (Poisson)! (e.g., see Odlyzko [53] and Sarnak [60]).

All these problems have led to three very natural random matrix ensembles: Hermitian, symmetric and symplectic ensembles. The purpose of this section is to show that these three examples appear very naturally as tangent spaces to symmetric spaces.

A symmetric space G/K is given by a semi-simple Lie group G and a Lie group involution $\sigma : G \to G$ such that

$$K = \{x \in G,\ \sigma(x) = x\}.$$

Then the following identification holds:

$$G/K \cong \{g\sigma(g)^{-1} \text{ with } g \in G\},$$

and the involution σ induces a map of the Lie algebra

$$\sigma_* : \mathfrak{g} \longrightarrow \mathfrak{g} \ \text{ such that } \ (\sigma_*)^2 = 1,$$

with

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \text{ with } \begin{cases} \mathfrak{k} = \{a \in \mathfrak{g} \text{ such that } \sigma_*(a) = a\} \\ \mathfrak{p} = \{a \in \mathfrak{g} \text{ such that } \sigma_*(a) = -a\} \end{cases}$$

and

$$[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k},\ [\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p},\ [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}.$$

Then K acts on $\mathfrak{p}$ by conjugation: $k\mathfrak{p}k^{-1} \subset \mathfrak{p}$ for all $k \in K$ and $\mathfrak{p}$ is the tangent space to G/K at the identity. The action of K on $\mathfrak{p}$ induces a root space decomposition, with $\mathfrak{a}$ being a maximal abelian subalgebra in $\mathfrak{p}$:

$$\mathfrak{p} = \mathfrak{a} + \sum_{\alpha \in \Delta} \mathfrak{p}_\alpha, \quad \text{with } m_\alpha = \dim \mathfrak{p}_\alpha.$$

Then, according to Helgason [35], the volume element on $\mathfrak{p}$ is given by

$$dV = \left(\prod_{\alpha \in \Delta_+} \alpha(z)^{m_\alpha} \right) dz_1 \ldots dz_n,$$

where Δ_+ is the set of positive roots; see [35, 36, 61, 62]. This will subsequently be worked out for the three so-called A_n-symmetric spaces. See also Sarnak's MSRI-lecture [59] in these proceedings, who deals with more general symmetric spaces. I like to thank Chuu-Lian Terng for very helpful conversations on these matters.

Examples:

(i) Hermitian ensemble

Consider the *non-compact symmetric space*[1] $SL(n,\mathbb{C})/SU(n)$ with $\sigma(g)=\bar{g}^{\top -1}$. Then

$$\begin{aligned} SL(n,\mathbb{C})/SU(n) &= \{g\bar{g}^{\top} \mid g\in SL(n,\mathbb{C})\} \\ &= \{\text{positive definite matrices with } \det = 1\} \end{aligned}$$

with

$$K=\{g\in SL(n,\mathbb{C})\mid \sigma(g)=g\}=\{g\in SL(n,\mathbb{C})\mid g^{-1}=\bar{g}^{\top}\}=SU(n).$$

Then $\sigma_*(a)=-\bar{a}^{\top}$ and the tangent space to G/K is then given by the space $\mathfrak{p}=\mathcal{H}_n$ of Hermitian matrices

$$sl(n,\mathbb{C})=\mathfrak{k}\oplus\mathfrak{p}=su(n)\oplus\mathcal{H}_n,\quad \text{i.e., } a=a_1+a_2,\ a_1\in su(n),\ a_2\in\mathcal{H}_n.$$

If $M\in\mathcal{H}_n$, then the M_{ii}, $\Re M_{ij}$ and $\Im M_{ij}$ $(1\le i<j\le n)$ are free variables, so that Haar measure on $M\in\mathcal{H}_n$ takes on the following form:

$$dM := \prod_1^n dM_{ii} \prod_{1\le i<j\le n} (d\Re M_{ij}\, d\Im M_{ij}). \tag{1.1.1}$$

A maximal abelian subalgebra $a\subset\mathfrak{p}=\mathcal{H}_n$ is given by real diagonal matrices $z=\operatorname{diag}(z_1,\dots,z_n)$. Each $M\in\mathfrak{p}=\mathcal{H}_n$ can be written as

$$M=e^{A}z\,e^{-A},\quad e^{A}\in K=SU(n),$$

with[2]

$$A=\sum_{1\le k\le\ell\le n}(a_{k\ell}(e_{k\ell}-e_{\ell k})+ib_{k\ell}(e_{k\ell}+e_{\ell k}))\in\mathfrak{k}=su(n),\quad a_{\ell\ell}=0. \tag{1.1.2}$$

Notice that $e_{k\ell}-e_{\ell k}$ and $i(e_{k\ell}+e_{\ell k})\in\mathfrak{k}=su(n)$ and that

$$[e_{k\ell}-e_{\ell k},z]=(z_\ell-z_k)(e_{k\ell}+e_{\ell k})\in\mathfrak{p}=\mathcal{H}_n$$

[1] The corresponding compact symmetric space is given by $(SU(n)\times SU(n))/SU(n)$.
[2] $e_{k\ell}$ is the $n\times n$ matrix with all zeroes, except for 1 at the (k,ℓ)th entry.

$$[i(e_{k\ell}+e_{\ell k}),z]=(z_\ell-z_k)i(e_{k\ell}-e_{\ell k})\in\mathfrak{p}=\mathcal{H}_n. \tag{1.1.3}$$

Incidentally, this implies that $e_{k\ell}+e_{\ell k}$ and $i(e_{k\ell}+e_{\ell k})$ are two-dimensional eigenspaces[3] of $(ad\,z)^2$ with eigenvalue $(z_\ell-z_k)^2$. From (1.1.2) and (1.1.3) it follows that

$$[A,z]=(z_\ell-z_k)\sum_{1\le k<\ell\le n}(a_{k\ell}(e_{k\ell}+e_{\ell k})+ib_{k\ell}(e_{k\ell}-e_{\ell k}))\in\mathfrak{p}=\mathcal{H}_n \tag{1.1.4}$$

and thus, for small A, we have[4]

$$\begin{aligned} dM &= d(e^A z\,e^{-A})\\ &= d(z+[A,z]+\dots)\\ &= \prod_1^n dz_i \prod_{1\le k<\ell\le n} d((z_\ell-z_k)a_{k\ell})d((z_\ell-z_k)b_{k\ell}),\ \text{using (1.1.4) and (1.1.1)}\\ &= \prod_1^n dz_i\Delta_n^2(z)\prod_{1\le k<\ell\le n} da_{k\ell}db_{k\ell}. \end{aligned} \tag{1.1.5}$$

Therefore $\Delta^2(z)$ is also the Jacobian determinant of the map $M\to(z,U)$, such that $M=UzU^{-1}\in\mathcal{H}_n$, and thus dM admits the decomposition in polar coordinates:

$$dM=\Delta_n^2(z)dz_1\dots dz_n dU,\quad U\in SU(n). \tag{1.1.6}$$

In random matrix theory, $\mathcal{H}_n$ is endowed with the following probability,

$$P(M\in dM)=c_n e^{-trV(M)}dM,\quad \rho(dz)=e^{-V(z)}dz, \tag{1.1.7}$$

where dM is Haar measure (1.1.6) on $\mathcal{H}_n$ and c_n is the normalizing factor. Since dM as in (1.1.6) contains dU and since the probability measure (1.1.7) only depends on the trace of $V(M)$, dU completely integrates out. Given $E\subset\mathbb{R}$, define

$$\mathcal{H}_n(E):=\{M\in\mathcal{H}_n\text{with all spectral points}\in E\subset\mathbb{R}\}\subset\mathcal{H}_n. \tag{1.1.8}$$

Then

$$P(M\in\mathcal{H}_n(E))=\int_{\mathcal{H}_n(E)}c_n e^{-\operatorname{Tr}V(M)}dM=\frac{\int_{E^n}\Delta^2(z)\prod_1^n\rho(dz_k)}{\int_{\mathbb{R}^n}\Delta^2(z)\prod_1^n\rho(dz_k)}. \tag{1.1.9}$$

As the reader can find out from the excellent book by Mehta [49], it is well known that, if the probability $P(M\in dM)$ satisfies the following two requirements: (i) invariance under conjugation by unitary transformations $M\mapsto UMU^{-1}$, (ii) the random variables M_{ii}, $\Re M_{ij}$, $\Im M_{ij}$, $1\le i<j\le n$ are independent, then $V(z)$ is quadratic (Gaussian ensemble).

[3] $ad\,x(y):=[x,y]$.

[4] $\Delta_n(z)=\prod_{1\le i<j\le n}(z_i-z_j)$ is the Vandermonde determinant.

(ii) Symmetric ensemble

Here we consider the *non-compact symmetric space*[5] $SL(n,\mathbb{R})/SO(n)$ with $\sigma(g) = g^{\top -1}$. Then

$$\begin{aligned} SL(n,\mathbb{R})/SO(n) &= \{gg^\top \mid g \in SL(n,\mathbb{R})\} \\ &= \{\text{positive definite matrices with } \det = 1\} \end{aligned}$$

with

$$K = \{g \in SL(n,\mathbb{R}) \mid \sigma(g) = g\} = \{g \in SL(n,\mathbb{R}) \mid g^\top = g^{-1}\} = SO(n).$$

Then $\sigma_*(a) = -a^\top$ and the tangent space to G/K is then given by the space $\mathfrak{p} = \mathcal{S}_n$ of symmetric matrices, appearing in the decomposition of $sl(n,\mathbb{R})$,

$$sl(n,\mathbb{R}) = \mathfrak{k} \oplus \mathfrak{p} = so(n) \oplus \mathcal{S}_n, \quad \text{i.e.,}\ a = a_1 + a_2,\ a_1 \in so(n),\ a_2 \in \mathcal{S}_n$$

with Haar measure $dM = \prod_{1 \le i \le j \le n} dM_{ij}$ on $\mathcal{S}_n$.

A maximal abelian subalgebra $a \subset \mathfrak{p} = \mathcal{S}_n$ is given by real traceless diagonal matrices $z = \operatorname{diag}(z_1, \dots, z_n)$. Each $M \in \mathfrak{p} = \mathcal{S}_n$ conjugates to a diagonal matrix z

$$M = e^A z\, e^{-A}, \quad e^A \in K = SO(n), \quad A \in so(n).$$

A calculation, analogous to example (i)(1.1.5) leads to

$$dM = \left|\Delta_n(z)\right| dz_1 \dots dz_n dU, \quad U \in SO(n).$$

Random matrix theory deals with the following probability on $\mathcal{S}_n$:

$$P(M \in dM) = c_n e^{-trV(M)} dM, \quad \rho(dz) = e^{-V(z)} dz, \tag{1.1.10}$$

with normalizing factor c_n. Setting as in (1.1.8): $\mathcal{S}_n(E) \subset \mathcal{S}_n$ is the subset of matrices with spectrum $\in E$. Then

$$P(M \in \mathcal{S}_n(E)) = \int_{\mathcal{S}_n(E)} c_n e^{-\operatorname{Tr} V(M)} dM = \frac{\int_{E^n} |\Delta(z)| \prod_1^n \rho(dz_k)}{\int_{\mathbb{R}^n} |\Delta(z)| \prod_1^n \rho(dz_k)}. \tag{1.1.11}$$

As in the Hermitian case, $P(M \in dM)$ is Gaussian, if $P(M \in dM)$ satisfies *(i)* invariance under conjugation by orthogonal conjugation $M \to OMO^{-1}$, *(ii)* M_{ii}, M_{ij} $(i < j)$ are independent random variables.

[5]The compact version is given by $SU(n)/SO(n)$.

(iii) Symplectic ensemble

Consider the *non-compact symmetric space*[6] $SU^*(2n)/USp(n)$ with $\sigma(g) = Jg^{\top -1}J^{-1}$, where J is the $2n \times 2n$ matrix:

$$J := \begin{pmatrix} \boxed{\begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix}} & & & \\ & \boxed{\begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix}} & & \\ & & \boxed{\begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix}} & \\ & & & \ddots \end{pmatrix} \quad \text{with } J^2 = -I, \tag{1.1.12}$$

and

$$G \;=\; SU^*(2n) = \{g \in SL(2n,\mathbb{C}) \mid g = J\bar{g}J^{-1}\},$$

$$\begin{aligned} K &= \{g \in SU^*(2n) \mid \sigma(g) = g\} := Sp(n,\mathbb{C}) \cap U(2n) \\ &= \{g \in SL(2n,\mathbb{C}) \bigm| g^\top Jg = J\} \cap \{g \in SL(2n,\mathbb{C}) \bigm| g^{-1} = \bar{g}^\top\} \\ &= \{g \in SL(2n,\mathbb{C}) \bigm| g^{-1} = \bar{g}^\top \text{and } g = J\bar{g}J^{-1}\} \\ &=: USp(n). \end{aligned}$$

Then, $\sigma_*(a) = -Ja^\top J^{-1}$ and

$$\begin{aligned} \mathfrak{k} &= \{a \in su^*(2n) \bigm| \sigma_*(a) = a\} = sp(n,\mathbb{C}) \cap u(2n) \\ &= \{a \in \mathbb{C}^{2n\times 2n} \bigm| a^\top = -\bar{a},\ a = J\bar{a}J^{-1}\} \end{aligned}$$

$$\begin{aligned} \mathfrak{p} &= \{a \in su^*(2n) \bigm| \sigma_*(a) = -a\} = su^*(2n) \cap iu(2n) \\ &= \{a \in \mathbb{C}^{2n\times 2n} \bigm| a^\top = \bar{a},\ a = J\bar{a}J^{-1}\} \\ &= \left\{ M = (M_{k\ell})_{1\le k,\ell\le n}, M_{k\ell} = \begin{pmatrix} M_{k\ell}^{(0)} & M_{k\ell}^{(1)} \\ & \\ -\bar{M}_{k\ell}^{(1)} & \bar{M}_{k\ell}^{(0)} \end{pmatrix} \text{ with } \ M_{\ell k} = \bar{M}_{k\ell}^\top \in \mathbb{C}^{2\times 2} \right\} \\ &\cong \{\text{self-dual } n \times n \text{ Hermitean matrices, with quaternionic entries}\} \\ &=: \mathcal{T}_{2n}. \end{aligned}$$

The condition on the 2×2 matrices $M_{k\ell}$ implies that $M_{kk} = M_k I$, with $M_k \in \mathbb{R}$ and the 2×2 identity I. Notice $USp(n)$ acts naturally by conjugation on the tangent space

[6] The corresponding compact symmetric space is $SU(2n)/Sp(n)$.

$\mathfrak{p}$ to G/K. Haar measure on $\mathcal{T}_{2n}$ is given by

$$dM = \prod_1^n dM_k \prod_{1\le k<\ell\le n} dM_{k\ell}^{(0)} d\bar{M}_{k\ell}^{(0)} dM_{k\ell}^{(1)} d\bar{M}_{k\ell}^{(1)}, \tag{1.1.13}$$

since these M_{ij} are the only free variables in the matrix $M \in \mathcal{T}_{2n}$. A maximal abelian subalgebra in $\mathfrak{p}$ is given by real diagonal matrices of the form $z = \operatorname{diag}(z_1, z_1, z_2, z_2, ..., z_n, z_n)$. Each $M \in \mathfrak{p} = \mathcal{T}_{2n}$ can be written as

$$M = e^A z e^{-A}, \quad e^A \in K = USp(n), \tag{1.1.14}$$

with $(a_{k\ell}, b_{k\ell}, c_{k\ell}, d_{k\ell} \in \mathbb{R})$

$$A = \sum_{1\le k\le \ell\le n} a_{k\ell}(e_{k\ell}^{(0)} - e_{\ell k}^{(0)}) + b_{k\ell}(e_{k\ell}^{(1)} + e_{\ell k}^{(1)}) + c_{k\ell}(e_{k\ell}^{(2)} - e_{\ell k}^{(2)}) + d_{k\ell}(e_{k\ell}^{(3)} + e_{\ell k}^{(3)}) \in \mathfrak{k} \tag{1.1.15}$$

in terms of the four 2×2 matrices [7]

$$e^{(0)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; e^{(1)} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \; e^{(2)} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \; e^{(3)} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Since

$$\begin{aligned} [e_{k\ell}^{(0)} - e_{\ell k}^{(0)}, z] &= (z_\ell - z_k)(e_{k\ell}^{(0)} + e_{\ell k}^{(0)}) \in \mathfrak{p} \\ [e_{k\ell}^{(1)} + e_{\ell k}^{(1)}, z] &= (z_\ell - z_k)(e_{k\ell}^{(1)} - e_{\ell k}^{(1)}) \in \mathfrak{p} \\ [e_{k\ell}^{(2)} - e_{\ell k}^{(2)}, z] &= (z_\ell - z_k)(e_{k\ell}^{(2)} + e_{\ell k}^{(2)}) \in \mathfrak{p} \\ [e_{k\ell}^{(3)} + e_{\ell k}^{(3)}, z] &= (z_\ell - z_k)(e_{k\ell}^{(3)} - e_{\ell k}^{(3)}) \in \mathfrak{p}, \end{aligned} \tag{1.1.16}$$

$[A, z] \in \mathfrak{p}$ has the following form: it has 2×2 zero blocks along the diagonal and from (1.1.16) and (1.1.15),

$$((k,\ell)\text{th block in } [A,z]) = (z_\ell - z_k)\begin{pmatrix} a_{k\ell} + ib_{k\ell} & c_{k\ell} + id_{k\ell} \\ -c_{k\ell} + id_{k\ell} & a_{k\ell} - ib_{k\ell} \end{pmatrix}, (k<\ell). \tag{1.1.17}$$

Therefore, using (1.1.17), Haar measure dM on $\mathcal{T}_{2n}$ equals

$$\begin{aligned} dM &= d(e^A z e^{-A}) \\ &= d(I + A + \dots)z(I - A + \dots) \\ &= d(z + [A,z] + \dots) \\ &= \prod_{1\le k\le n} dz_k \prod_{1\le k<\ell\le n} d((z_\ell - z_k)(a_{k\ell} + ib_{k\ell}))d((z_\ell - z_k)(a_{k\ell} - ib_{k\ell})) \\ &\qquad d((z_\ell - z_k)(c_{k\ell} + id_{k\ell}))d((z_\ell - z_k)(-c_{k\ell} + id_{k\ell})) \\ &= \Delta^4(z) dz_1 \cdots dz_n \prod_{1\le k<\ell\le n} 4 da_{k\ell} db_{k\ell} dc_{k\ell} dd_{k\ell}. \end{aligned}$$

[7] $e_{k\ell}^{(i)}$ in (1.1.15) refers to putting the 2×2 matrix $e^{(i)}$ at place (k,ℓ).

As before, define $\mathcal{T}_{2n}(E) \subset \mathcal{T}_{2n}$ as the subset of matrices with spectrum $\in E$ and define the probability:

$$P(M \in \mathcal{T}_{2n}(E)) = \int_{\mathcal{T}_{2n}(E)} c_n e^{-\operatorname{Tr} V(M)} dM = \frac{\int_{E^n} \Delta^4(z) \prod_1^n \rho(dz_k)}{\int_{\mathbb{R}^n} \Delta^4(z) \prod_1^n \rho(dz_k)}. \tag{1.1.18}$$

<u>*Remark:*</u> Notice $\mathcal{T}_{2n}$ is called the symplectic ensemble, although the matrices in $\mathfrak{p} = \mathcal{T}_{2n}$ are not at all symplectic; but rather the matrices in $\mathfrak{k}$ are.

1.2 Infinite Hermitian matrix ensembles

Consider now he limit of the probability

$$P(M \in \mathcal{H}_n(E)) = \frac{\int_{E^n} \Delta^2(z) \prod_1^n \rho(dz_k)}{\int_{\mathbb{R}^n} \Delta^2(z) \prod_1^n \rho(dz_k)}, \text{ when } n \nearrow \infty. \tag{1.2.1}$$

Dyson [27] (see also Mehta [49]) used the following trick, to circumvent the problem of dealing with ∞-fold integrals. Using the orthogonality of the *monic orthogonal polynomials* $p_k = p_k(z)$ for the weight $\rho(dz)$ on $\mathbb{R}$, and the L^2-norms $h_k = \int_{\mathbb{R}} p_k^2(z)\rho(dz)$ of the p_k's, one finds, using $(\det A)^2 = \det(AA^\top)$,

$$\begin{aligned}
&\int_{\mathbb{R}^n} \Delta^2(z) \prod_1^n \rho(dz_i) \\
&\quad = \int_{\mathbb{R}^n} \det(p_{i-1}(z_j))_{1\le i,j\le n} \det(p_{k-1}(z_\ell))_{1\le k,\ell\le n} \prod_{k=1}^n \rho(dz_k) \\
&\quad = \sum_{\pi,\pi'\in\sigma_n} (-1)^{\pi+\pi'} \prod_{k=1}^n \int_{\mathbb{R}} p_{\pi(k)-1}(z_k) p_{\pi'(k)-1}(z_k)\rho(dz_k) \\
&\quad = n! \prod_0^{n-1} \int_{\mathbb{R}} p_k^2(z)\rho(dz) = n! \prod_0^{n-1} h_k.
\end{aligned} \tag{1.2.2}$$

For the integral over an arbitrary subset $E \subset \mathbb{R}$, one stops at the second equality, since the p_n's are not necessarily orthogonal over E. This leads to the probability (1.2.1),

$$\begin{aligned}
&P(M \in \mathcal{H}_n(E))) \\
&\quad = \frac{1}{n! \prod_1^n h_{i-1}} \int_{E^n} \det\left(\sum_{1\le j\le n} p_{j-1}(z_k)p_{j-1}(z_\ell)\right)_{1\le k,\ell\le n} \prod_1^n \rho(dz_i) \\
&\quad = \frac{1}{n!} \int_{E^n} \det(K_n(z_k,z_\ell))_{1\le k,\ell\le n} \prod_1^n \rho(dz_i),
\end{aligned} \tag{1.2.3}$$

in terms of the kernel

$$K_n(y,z) := \sum_{j=1}^{n} \frac{p_{j-1}(y)}{\sqrt{h_{j-1}}} \frac{p_{j-1}(z)}{\sqrt{h_{j-1}}}. \tag{1.2.4}$$

The orthonormality relations of the $p_k(y)/\sqrt{h_k}$ lead to the reproducing property for the kernel $K_n(y,z)$:

$$\int_{\mathbb{R}} K_n(y,z)K_n(z,u)\rho(dz) = K_n(y,u), \qquad \int_{\mathbb{R}} K_n(z,z)\rho(dz) = n. \tag{1.2.5}$$

Upon replacing E^n by $\prod_1^k dz_i \times \mathbb{R}^{n-k}$ in (1.2.3), upon integrating out all the remaining variables $z_{k+1},...,z_n$ and using the reproducing property (1.2.5), one finds the n-point correlation function

$$\begin{aligned} & P(\text{one eigenvalue in each } [z_i, z_i+dz_i],\ i=1,...,k) \\ = \quad & c_n \det\left(K_n(z_i,z_j)\right)_{1\le i,j\le k} \prod_1^k \rho(dz_i). \end{aligned} \tag{1.2.6}$$

Finally, by Poincaré's formula for the probability $P\big(\cup E_i\big)$, the probability that no spectral point of M belongs to E is given by a Fredholm determinant

$$\begin{aligned} P(M\in\mathcal{H}_n(E^c)) &= \det(I-\lambda K_n^E) \\ &= 1+\sum_{k=1}^{\infty}(-\lambda)^k \int_{z_1\le\ldots\le z_k} \det\Big(K_n^E(z_i,z_j)\Big)_{1\le i,j\le k} \prod_1^k \rho(dz_i), \end{aligned}$$

for the kernel $K_n^E(y,z) = K_n(y,z)I_E(z)$.

• *Wigner's semi-circle law*: For this ensemble (defined by a large class of ρ's, in particular for the Gaussian ensemble) and for very large n, the density of eigenvalues tends to Wigner's semi-circle distribution on the interval $[-\sqrt{2n},\sqrt{2n}]$:

$$\text{density of eigenvalues} \begin{cases} = \dfrac{1}{\pi}\sqrt{2n-z^2}\,dz, & |z|\le\sqrt{2n} \\[2ex] = 0, & |z|>\sqrt{2n}. \end{cases}$$

• *Bulk scaling limit*: From the formula above, it follows that the average number of eigenvalues per unit length near $z=0$ ("the bulk") is given by $\sqrt{2n}/\pi$ and thus the average distance between two consecutive eigenvalues is given by $\pi/\sqrt{2n}$. Upon using this rescaling, one shows ([43, 48, 52, 55, 39])

$$\lim_{n\nearrow\infty} \frac{\pi}{\sqrt{2n}} K_n\left(\frac{\pi x}{\sqrt{2n}}, \frac{\pi y}{\sqrt{2n}}\right) = \frac{\sin\pi(x-y)}{\pi(x-y)} \qquad \text{(Sine kernel)}$$

and

$$P(\text{exactly } k \text{ eigenvalues } \in [0,a]) = \frac{(-1)^k}{k!}\left(\frac{\partial}{\partial\lambda}\right)^k \det(I - \lambda K I_{[0,a]})\Big|_{\lambda=1}$$

with

$$\det(I - \lambda K I_{[0,a]}) = \exp \int_0^{\pi a} \frac{f(x;\lambda)}{x} dx, \tag{1.2.7}$$

where $f(x,\lambda)$ is a solution to the following differential equation, due to the pioneering work of Jimbo, Miwa, Mori, Sato [39], ($' = \partial/\partial x$)

$$(xf'')^2 \;=\; 4(xf'-f)(-f'^2 - xf' + f)\ ,\ \text{with } f(x;\lambda) \cong -\frac{\lambda}{\pi}x \text{ for } x \simeq 0. \qquad (\textbf{Painlevé V}) \tag{1.2.8}$$

• *Edge scaling limit*: Near the edge $\sqrt{2n}$ of the Wigner semi-circle, the scaling is $\sqrt{2}n^{1/6}$ and thus the scaling is more subtle: (see [21, 30, 51, 49, 64])

$$y = \sqrt{2n} + \frac{u}{\sqrt{2}n^{1/6}}, \tag{1.2.9}$$

and so for the kernel K_n as in (1.2.4), with the p_n's being Hermite polynomials,

$$\lim_{n\nearrow\infty} \frac{1}{\sqrt{2}n^{1/6}} K_n\left(\sqrt{2n} + \frac{u}{\sqrt{2}n^{1/6}}, \sqrt{2n} + \frac{v}{\sqrt{2}n^{1/6}}\right) = K(u,v),$$

where

$$K(u,v) = \int_0^\infty A(x+u)A(x+v)dx, \quad A(u) = \int_{-\infty}^{\infty} e^{iux - x^3/3}dx.$$

Relating y and u by (1.2.9), the statistics of the largest eigenvalue for very large n is governed by the function,

$$\begin{aligned} P(\lambda_{\max} \le y) \;&=\; P\left(2n^{\frac{2}{3}}\left(\frac{\lambda_{\max}}{\sqrt{2n}} - 1\right) \le u\right),\ \text{for } n \nearrow \infty, \\ &=\; \det(I - K I_{(-\infty,u]}) = \exp\left(-\int_u^\infty (\alpha - u) g^2(\alpha) d\alpha\right), \end{aligned}$$

with $g(x)$ a solution of

$$\begin{cases} g'' = xg + 2g^3 \\ g(x) \cong -\frac{e^{-\frac{2}{3}x^{\frac{3}{2}}}}{2\sqrt{\pi}x^{1/4}} \text{ for } x \nearrow \infty. \end{cases} \qquad (\textbf{Painlevé II}) \tag{1.2.10}$$

The latter is essentially the asymptotics of the Airy function. In section 5, I shall derive, via Virasoro constraints, not only this result, due to Tracy-Widom [64], but

also a PDE for the probability that the eigenvalues belong to several intervals, due to Adler-Shiota-van Moerbeke [11, 12].

• *Hard edge scaling limit*: Consider the ensemble of $n \times n$ random matrices for the Laguerre probability distribution, thus corresponding to (1.1.9) with $\rho(dz) = z^{\nu/2}e^{-z/2}dz$. One shows the density of eigenvalues near $z = 0$ is given by $4n$ for very large n. At this edge, one computes for the kernel (1.2.4) with Laguerre polynomials p_n [52, 30]:

$$\lim_{n\nearrow\infty} \frac{1}{4n} K_n^{(\nu)}\left(\frac{u}{4n}, \frac{v}{4n}\right) = K^{(\nu)}(u,v), \tag{1.2.11}$$

where $K^{(\nu)}(u,v)$ is the *Bessel kernel*, with Bessel functions J_ν,

$$\begin{aligned} K^{(\nu)}(u,v) &= \frac{1}{2}\int_0^1 xJ_\nu(xu)J_\nu(xv)dx \\ &= \frac{J_\nu(u)\sqrt{u}J'_\nu(v) - J_\nu(\sqrt{v})\sqrt{v}J'_\nu(\sqrt{u})}{2(u-v)}. \end{aligned} \tag{1.2.12}$$

Then

$$P(\text{ no eigenvalues } \in [0,x]) = \exp\left(-\int_0^x \frac{f(u)}{u}du\right),$$

with f satisfying

$$(xf'')^2 - 4\big(xf' - f\big)f'^2 + \big((x-\nu^2)f' - f\big)f' = 0. \ \textbf{(Painlevé V)} \tag{1.2.13}$$

This result due to Tracy-Widom [65] and a more general statement, due to [11, 12] will be shown using Virasoro constraints in section 5.

1.3 Integrals over classical groups

The integration on a compact semi-simple simply connected Lie group G is given by the formula (see Helgason [36])

$$\int_G f(M)dM = \frac{1}{|W|}\int_T \left|\prod_{\alpha\in\Delta} 2\sin\frac{\alpha(iH)}{2}\right| dt \int_U f(utu^{-1})du, \quad t = e^H, \tag{1.3.1}$$

where $A \subset G$ is a maximal subgroup, with $\mathfrak{g}$ and $\mathfrak{a}$ being the Lie algebras of G and A. Let du and dt be Haar measures on G, A respectively such that

$$\int_A dt = \int_U du = 1;$$

Δ denotes the set of roots of $\mathfrak{g}$ with respect to $\mathfrak{a}$; $|W|$ is the order of the Weyl group of G.

Integration formula (1.3.1) will be applied to integrals of $f=e^{\sum_1^\infty t_i \operatorname{Tr} M^i}$ over the groups $SO(2n)$, $SO(2n{+}1)$ and $Sp(n)$. Their Lie algebras (over $\mathbb{C}$) are given respectively by $\mathfrak{d}_n$, $\mathfrak{b}_n$, $\mathfrak{c}_n$, with sets of roots: (e.g., see [20])

$$\Delta_n=\{\pm\varepsilon e_i, 1\le i\le k, \pm(e_i+e_j), \pm(e_i-e_j), 1\le i<j\le n\},$$

with

$$\begin{array}{l} \varepsilon=0 \text{ for } \mathfrak{d}_n=so(2n)\\ \varepsilon=1 \text{ for } \mathfrak{b}_n=so(2n+1)\\ \varepsilon=2 \text{ for } \mathfrak{c}_n=sp(n). \end{array}$$

Setting $H=i\theta$, we have, in view of formula (1.3.1),

$$\begin{aligned}
&\left|\prod_{\alpha\in\Delta} 2\sin\frac{\alpha(iH)}{2}\right| dt\\
&\quad= \left\{\begin{array}{ll} c_n\left(\prod_{1\le j<k\le n}\sin\frac{\theta_j-\theta_k}{2}\sin\frac{\theta_j+\theta_k}{2}\right)^2\prod_1^n d\theta_j & \text{for } \mathfrak{d}_n\\ c_n\left(\prod_{1\le j<k\le n}\sin\frac{\theta_j-\theta_k}{2}\sin\frac{\theta_j+\theta_k}{2}\right)^2\prod_1^n \sin^2\frac{\varepsilon\theta_j}{2}d\theta_j & \text{for } \mathfrak{b}_n,\ \mathfrak{c}_n \end{array}\right.\\
&\quad= c_n'\prod_{1\le j<k'\le n}(\cos\theta_j-\cos\theta_k)^2\left\{\begin{array}{ll} \displaystyle\prod_{1\le j\le n} d\theta_j & \text{for } \mathfrak{d}_n\\ \displaystyle\prod_{1\le j\le n}\left(\frac{1-\cos\theta_j}{2}\right)d\theta_j & \text{for } \mathfrak{b}_n\\ \displaystyle\prod_{1\le j\le n}(1-\cos^2\theta_j)d\theta_j & \text{for } \mathfrak{c}_n \end{array}\right.\\
&\quad= \left\{\begin{array}{ll} \displaystyle c_n'\Delta^2(z)\prod_{1\le j\le n}\frac{dz_j}{\sqrt{1-z_j^2}} & \text{for } \mathfrak{d}_n\\ \displaystyle c_n'\Delta^2(z)\prod_{1\le j\le n}(1-z_j)\frac{dz_j}{\sqrt{1-z_j^2}} & \text{for } \mathfrak{b}_n\\ \displaystyle c_n'\Delta^2(z)\prod_{1\le j\le n}(1-z_j^2)\frac{dz_j}{\sqrt{1-z_j^2}} & \text{for } \mathfrak{c}_n \end{array}\right.\\
&\quad= c_n''\Delta^2(z)\prod_{1\le j\le n}(1-z_j)^\alpha(1+z_j)^\beta dz_j \text{ with } \left\{\begin{array}{ll} \alpha=\beta=-1/2 & \text{for } \mathfrak{d}_n\\ \alpha=1/2, \beta=-1/2 & \text{for } \mathfrak{b}_n\\ \alpha=\beta=1/2 & \text{for } \mathfrak{c}_n \end{array}\right.
\end{aligned}$$

For $M\in SO(2n)$, $Sp(n)$, the eigenvalues are given by $e^{i\theta_j}$ and $e^{-i\theta_j}$, $1\le j\le n$; therefore, setting $f=\exp(\sum t_k trM^k)$ in formula (1.3.1), leads to

$$e^{\sum_1^\infty t_k \operatorname{Tr} M^k}=e^{\sum_1^\infty t_k\sum_{j=1}^n(e^{ik\theta_j}+e^{-ik\theta_j})}=\prod_{j=1}^n e^{2\sum_{k=1}^\infty t_k\cos k\theta_j}=\prod_{j=1}^n e^{2\sum t_kT_k(z_j)}, \qquad (1.3.2)$$

where $T_n(z)$ are the Tchebychev polynomials, defined by $T_n(\cos\theta) := \cos n\theta$; in particular $T_1(z) = z$.

For $M \in SO(2n+1)$, the eigenvalues are given by 1, $e^{i\theta_j}$ and $e^{-i\theta_j}$, $1 \le j \le n$, which is responsible for the extra-exponential $e^{\sum t_i}$ appearing in (1.3.2).

Before listing various integrals, define the Jacobi weight

$$\rho_{\alpha\beta}(z)dz := (1-z)^{\alpha}(1+z)^{\beta}dz, \tag{1.3.3}$$

and the formal sum

$$g(z) := 2\sum_1^{\infty} t_i T_i(z).$$

The arguments above lead to the following integrals, originally due to H. Weyl [75], and in its present form, due to Johansson [40]; besides the integrals over $SO(k) = O_+(k)$, the integrals over $O_-(k)$ and $U(n)$ will also be of interest in the theory of random permutations:

$$\begin{aligned}
\int_{O(2n)_+} e^{\sum_1^\infty t_i trM^i} dM &= \int_{[-1,1]^n} \Delta_n(z)^2 \prod_{k=1}^n e^{g(z_k)} \rho_{(-\frac{1}{2},-\frac{1}{2})}(z_k)dz_k \\
\int_{O(2n+1)_+} e^{\sum_1^\infty t_i trM^i} dM &= e^{\sum_1^\infty t_i} \int_{[-1,1]^n} \Delta_n(z)^2 \prod_{k=1}^n e^{g(z_k)} \rho_{(\frac{1}{2},-\frac{1}{2})}(z_k)dz_k \\
\int_{Sp(n)} e^{\sum_1^\infty t_i trM^i} dM &= \int_{[-1,1]^n} \Delta_n(z)^2 \prod_{k=1}^n e^{g(z_k)} \rho_{(\frac{1}{2},\frac{1}{2})}(z_k)dz_k. \\
\int_{O(2n)_-} e^{\sum_1^\infty t_i trM^i} dM &= e^{\sum_1^\infty 2t_{2i}} \int_{[-1,1]^{n-1}} \Delta_{n-1}(z)^2 \prod_{k=1}^{n-1} e^{g(z_k)} \rho_{(\frac{1}{2},\frac{1}{2})}(z_k)dz_k \\
\int_{O(2n+1)_-} e^{\sum_1^\infty t_i trM^i} dM &= e^{\sum_1^\infty (-1)^i t_i} \int_{[-1,1]^n} \Delta_n(z)^2 \prod_{k=1}^n e^{g(z_k)} \rho_{(-\frac{1}{2},\frac{1}{2})}(z_k)dz_k \\
\\
\int_{U(n)} e^{\sum_1^\infty tr(t_i M^i - s_i \bar M^i)} dM &= \frac{1}{n!}\int_{(S^1)^n} |\Delta_n(z)|^2 \prod_{k=1}^n e^{\sum_1^\infty (t_i z_k^i - s_i z_k^{-i})} \frac{dz_k}{2\pi i z_k}
\end{aligned} \tag{1.3.4}$$

1.4 Permutations and integrals over groups

Let S_n be the group of permutations π and S_{2n}^0 the subset of fixed-point free involutions π^0(i.e., $(\pi^0)^2 = I$ and $\pi^0(k) \ne k$ for $1 \le k \le 2n$). Put the uniform distribution on S_n and S_{2n}^0 ; i.e., all permutations or involutions have equal probability:

$$P(\pi_n) = 1/n! \;\text{ and }\; P(\pi_{2n}^0) = \frac{2^n n!}{(2n)!}; \tag{1.4.1}$$

π_n refers to a permutation in S_n and π^0_{2n} to an involution in S^0_{2n}.

An *increasing subsequence* of $\pi \in S_n$ or S^0_n is a sequence $1 \leq j_1 < ... < j_k \leq n$, such that $\pi(j_1) < ... < \pi(j_k)$. Define

$$L(\pi_n) = \text{ length of the longest increasing subsequence of } \pi_n \ . \tag{1.4.2}$$

Example: for $\pi = (\underline{3}, 1, \underline{4}, 2, \underline{6}, \underline{7}, 5)$, we have $L(\pi_7) = 4$.

Around 1960 and based on Monte-Carlo methods, Ulam [70] conjectured that

$$\lim_{n\to\infty} \frac{E(L_n)}{\sqrt{n}} = c \ \text{ exists.}$$

An argument of Erdös & Szekeres [28], dating back from 1935 showed that $E(L_n) \geq \frac{1}{2}\sqrt{n-1}$, and thus $c \geq 1/2$. In '72, Hammersley [33] showed rigorously that the limit exists. Logan and Shepp [46] showed the limit $c \geq 2$, and finally Vershik and Kerov [74] that $c = 2$. In 1990, I. Gessel [31] showed that the following generating function is the determinant of a Toeplitz matrix:

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} P(L_n \leq \ell) = \det \left(\int_0^{2\pi} e^{2\sqrt{t}\cos\theta} e^{i(k-m)\theta} d\theta \right)_{0\leq k,m\leq \ell-1} . \tag{1.4.3}$$

The next major contribution was due to Johansson [41] and Baik-Deift-Johansson [17], who prove that for arbitrary $x \in \mathbb{R}$, we have a "*law of large numbers*" and a "*central limit theorem*", where $F(x)$ is the statistics (1.2.10),

$$\lim_{n\to\infty} \frac{L_n}{2\sqrt{n}} = 1, \quad \text{and} \quad P\left(\frac{L_n - 2\sqrt{n}}{n^{1/6}} \leq x\right) \longrightarrow F(x), \qquad \text{for } n \longrightarrow \infty.$$

A next set of ideas is due to Diaconis & Shashahani [26], Rains [57, 58], Baik & Rains [18]. For a nice state-of-the-art account, see Aldous & Diaconis [14]. An illustration is contained in the following proposition; the first statement is essentially Gessel's and the next statement is due to [26, 58, 18].

Proposition 1.1 *The following holds*

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{t^n}{n!} P(L(\pi_n) \leq \ell) &= \int_{U(\ell)} e^{\sqrt{t}\,\mathrm{Tr}(M+\bar{M})} dM \qquad (1.4.4) \\ &= \int_{[0,2\pi]^\ell} \prod_{1\leq j<k\leq \ell} |e^{i\theta_j} - e^{i\theta_k}|^2 \prod_{1\leq k\leq \ell} e^{2\sqrt{t}\cos\theta_k} \frac{d\theta_k}{2\pi}. \end{aligned}$$

$$\sum_{n=0}^{\infty} \frac{(t^2/2)^n}{n!} P(L(\pi^0_{2n}) \leq \ell) = \int_{O(\ell)} e^{t\,\mathrm{Tr}\, M} dM. \tag{1.4.5}$$

The proof of this statement will be sketched later. The connection with integrable systems goes via the following chain of ideas:

$$\begin{array}{c}\text{Combinatorics}\\ \downarrow \\ \text{Robinson-Schensted-Knuth correspondence}\\ \downarrow \\ \text{Theory of symmetric polynomials}\\ \downarrow \\ \text{Integrals over classical groups}\\ \downarrow \\ \text{Integrable systems}\end{array}$$

All the arrows, but the last one, will be explained in this section; the last arrow will be discussed in sections 7 and 8. We briefly sketch a few of the basic well known facts going into these arguments. They can be found in MacDonald [47], Knuth [45], Aldous-Diaconis [14]. Useful facts on symmetric functions, applicable to integrable theory, can be found in the appendix to [1]. Let me mention a few of these facts:

- A *Young diagram* λ is a finite sequence of non-increasing, non-negative integers $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_\ell \geq 0$; also called a *partition* of $n = |\lambda| := \lambda_1 + ... + \lambda_\ell$, with $|\lambda|$ being the weight. It can be represented by a diagram, having λ_1 boxes in the first row, λ_2 boxes in the second row, etc..., all aligned to the left. A *dual Young diagram* $\hat{\lambda} = (\hat{\lambda}_1 \geq \hat{\lambda}_2 \geq ...)$ is the diagram obtained by flipping the diagram λ about its diagonal.

- A *Young tableau* of shape λ is an array of positive integers a_{ij} (at place (i,j) in the Young diagram) placed in the Young diagram λ, which are non-decreasing from left to right *and* strictly increasing from top to bottom.

- A *standard Young tableau* of shape λ is an array of integers $1, ..., n$ placed in the Young diagram, which are strictly increasing from left to right *and* from top to bottom. The number of Young tableaux of a given shape $\lambda = (\lambda_1 \geq ... \geq \lambda_m)$ is

given by a number of formulae (for the Schur polynomial s_λ, see below)[8]

$$\begin{aligned} f^\lambda &= \#\{\text{standard tableaux of shape } \lambda\} \\ &= \text{coefficient of } x_1 x_2 \ldots x_n \text{ in } s_\lambda(x) \\ &= \frac{|\lambda|!}{\prod_{\text{all } i,j} h_{ij}^\lambda} = |\lambda|! \det\left(\frac{1}{(\lambda_i - i + j)!}\right) \qquad (1.4.6) \\ &= |\lambda|! \prod_{1\le i<j\le m} (h_i - h_j) \prod_1^m \frac{1}{h_i!}, \quad \text{with } h_i := \lambda_i - i + m,\ m := \hat{\lambda}_1. \end{aligned}$$

- The *Schur polynomial* s_λ associated with a Young diagram λ is a symmetric function in the variables $x_1, x_2, \ldots$ (finite or infinite), defined by

$$s_\lambda(x_1, x_2, \ldots) := \sum_{\{a_{ij}\} \text{ tableaux of } \lambda} \prod_{ij} x_{a_{ij}}. \qquad (1.4.7)$$

- The linear *space* Λ_n *of symmetric polynomials* in $x_1, \ldots, x_n$ with rational coefficients comes equipped with the inner product

$$\begin{aligned} \langle f, g\rangle &= \frac{1}{n!} \int_{(S_1)^n} f(z_1, \ldots, z_n) g(\bar{z}_1, \ldots, \bar{z}_n) \prod_{1\le k<\ell\le n} |z_k - z_\ell|^2 \prod_1^n \frac{dz_k}{2\pi i z_k} \\ &= \int_{U(n)} f(M) g(\bar{M}) dM. \qquad (1.4.8) \end{aligned}$$

- An *orthonormal basis of the space* Λ_n is given by the Schur polynomials $s_\lambda(x_1, \ldots, x_n)$, in which the numbers a_{ij} are restricted to $1, \ldots, n$. Therefore, each symmetric function admits a "*Fourier series*"

$$f(x_1, \ldots, x_n) = \sum_{\substack{\lambda \text{ with} \\ \hat{\lambda}_1 \le n}} \langle f, s_\lambda\rangle\ s_\lambda(x_1, \ldots, x_n), \text{ with } \langle s_\lambda, s_{\lambda'}\rangle = \delta_{\lambda\lambda'}. \qquad (1.4.9)$$

In particular, one proves (see (1.4.6) for the definition of f^λ)

$$(x_1 + \ldots + x_n)^k = \sum_{\substack{|\lambda|=k \\ \hat{\lambda}_1 \le n}} f^\lambda s_\lambda, \qquad (1.4.10)$$

If $\lambda = (\lambda_1 \ge \ldots \ge \lambda_\ell > 0)$, with[9] $\hat{\lambda}_1 = \ell > n$, then obviously $s_\lambda = 0$.

[8] $h_{ij}^\lambda := \lambda_i + \hat{\lambda}_j - i - j + 1$ is the *hook length* of the i,jth box in the Young diagram; i.e., the length of the hook formed by drawing a horizontal line emanating from the center of the box to the right and a vertical line emanating from the center of the box to the bottom of the diagram.

[9] Remember, from the definition of the dual Young diagram, that $\hat{\lambda}_1 =$ the length of the first column of λ

- *Robinson-Schensted-Knuth correspondence*: There is a 1-1 correspondence

$$S_n \longrightarrow \left\{ \begin{array}{l} (P,Q), \text{two standard Young} \\ \text{tableaux from } 1,\dots,n, \text{ where} \\ P \text{ and } Q \text{ have the same shape} \end{array} \right\}$$

Given a permutation $i_1, ..., i_n$, the correspondence constructs two standard Young tableaux P, Q having the same shape λ. This construction is inductive. Namely, having obtained two equally shaped Young diagrams P_k, Q_k from $i_1, ..., i_k$, with the numbers $(i_1, ..., i_k)$ in the boxes of P_k and the numbers $(1, ..., k)$ in the boxes of Q_k, one creates a new diagram Q_{k+1}, by putting the *next number* i_{k+1} *in the first row of* P, according to the following rule:

(i) if $i_{k+1} \geq$ all numbers appearing in the first row of P_k, then one creates a new box with i_{k+1} in that box to the right of the first column,

(ii) if not, place i_{k+1} in the box (of the first row) with the smallest number higher than i_{k+1}. That number then gets pushed down to the second row of P_k according to the rule (i) or (ii), as if the first row had been removed.

The diagram Q is a bookkeeping device; namely, add a box (with the number $k+1$ in it) to Q_k exactly at the place, where the new box has been added to P_k. This produces a new diagram Q_{k+1} of same shape as P_{k+1}.

The inverse of this map is constructed essentially by reversing the steps above.

Example: $\pi = (5, 1, 4, 3, 2) \in S_5$,

$$\begin{array}{l} 5 \end{array} \qquad \begin{array}{l} 1 \\ 5 \end{array} \qquad \begin{array}{ll} 1 & 4 \\ 5 & \end{array} \qquad \begin{array}{ll} 1 & 3 \\ 4 & \\ 5 & \end{array} \qquad \begin{array}{ll} 1 & 2 \\ 3 & \\ 4 & \\ 5 & \end{array}$$

$$\begin{array}{l} 1 \end{array} \qquad \begin{array}{l} 1 \\ 2 \end{array} \qquad \begin{array}{ll} 1 & 3 \\ 2 & \end{array} \qquad \begin{array}{ll} 1 & 3 \\ 2 & \\ 4 & \end{array} \qquad \begin{array}{ll} 1 & 3 \\ 2 & \\ 4 & \\ 5 & \end{array}$$

$$\text{Hence } \pi \longrightarrow (P(\pi), Q(\pi)) = \left(\begin{pmatrix} 1 & 2 \\ 3 & \\ 4 & \\ 5 & \end{pmatrix}, \begin{pmatrix} 1 & 3 \\ 2 & \\ 4 & \\ 5 & \end{pmatrix} \right)$$

and so $L_5(\pi) = 2 = \#$columns of P or Q.

The Robinson-Schensted-Knuth correspondence has the following properties

- $\pi \mapsto (P, Q)$, then $\pi^{-1} \mapsto (Q, P)$

- length (longest increasing subsequence of π) = # (columns in P)
- length (longest decreasing subsequence of π) = # (rows in P)
- $\pi^2 = I$, then $\pi \mapsto (P, P)$
- $\pi^2 = I$, with k fixed points, then P has exactly k columns of odd length. (1.4.11)

From representation theory (see Weyl [75] and especially Rains [57]), one proves:

Lemma 1.2 *The following perpendicularity relations hold:*

(i) $\displaystyle\int_{U(n)} s_\lambda(M) s_\mu(\bar{M}) dM = \langle s_\lambda, s_\mu \rangle = \delta_{\lambda\mu}$

$$\begin{aligned} (ii) \int_{O(n)} s_\lambda(M) dM &= 1 \quad \textit{for } \lambda = (\lambda_1 \geq \ldots \geq \lambda_k \geq 0),\ k \leq n,\ \ \lambda_i \textit{ even} \\ &= 0 \quad \textit{otherwise.} \end{aligned}$$

$$\begin{aligned} (iii) \int_{Sp(n)} s_\lambda(M) dM &= 1 \quad \textit{for } \hat{\lambda}_i \textit{ even, } \hat{\lambda}_1 \leq 2n\ , \\ &= 0 \quad \textit{otherwise.} \end{aligned} \tag{1.4.12}$$

Proof of Proposition 1.1 : On the one hand,

$$\begin{aligned} &\langle (x_1 + \ldots + x_n)^k, (x_1 + \ldots + x_n)^k \rangle \\ &\quad = \sum_{\substack{|\lambda| = |\mu| = k \\ \hat{\lambda}_1, \hat{\mu}_1 \leq n}} f^\lambda f^\mu \langle s_\lambda, s_\mu \rangle \\ &\quad = \sum_{\substack{|\lambda| = k \\ \hat{\lambda}_1 \leq n}} (f^\lambda)^2 \\ &\quad = \sum_{\substack{|\lambda| = k \\ \lambda_1 \leq n}} (f^\lambda)^2 \\ &\quad = \#\{(P, Q),\ \text{standard Young tableaux, each of arbitrary shape } \lambda \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{with } |\lambda| = k,\ \lambda_1 \leq n\} \\ &\quad = \#\{\pi_k \in S_k \text{ such that } L(\pi_k) \leq n\}. \end{aligned} \tag{1.4.13}$$

On the other hand, notice that, upon setting $\theta_j = \theta_j' + \theta_1$ for $2 \leq j \leq n$, the expression $\displaystyle\prod_{1 \leq j < k \leq n} |e^{i\theta_j} - e^{i\theta_k}|^2$ is independent of θ_1. Then, setting $z_k = e^{i\theta_k}$, one computes:

$$
\begin{aligned}
&\langle (x_1+\ldots+x_n)^k, (x_1+\ldots+x_n)^\ell\rangle \\
&\quad = \frac{1}{n!}\int_{[0,2\pi]^n} (z_1+\ldots+z_n)^k(\bar z_1+\ldots+\bar z_n)^\ell \prod_{1\le j<k\le n} |e^{i\theta_j}-e^{i\theta_k}|^2 d\theta_1\ldots d\theta_n \\
&\quad = \frac{1}{n!}\int_{[0,2\pi]^n} e^{ik\theta_1}(1+z_2'+\ldots+z_n')^k e^{-i\ell\theta_1}(1+\bar z_2'+\ldots+\bar z_n')^\ell \\
&\qquad\qquad\qquad\qquad\qquad \prod_{1\le j<k\le n} |e^{i\theta_j}-e^{i\theta_k}|^2 d\theta_1\ldots d\theta_n \\
&\qquad\qquad\qquad \text{upon setting } \theta_j=\theta_j'+\theta_1, \quad \text{for } j\ge 2 \text{ and } z_k'=e^{i\theta_k'}, \\
&\quad = \frac{1}{n!}\int_0^{2\pi} e^{i(k-\ell)\theta_1} d\theta_1 \quad \times \quad (\text{an } \ n-1\text{-fold integral}) \\
&\quad = \delta_{k\ell}\langle (x_1+\ldots+x_n)^k, (x_1+\ldots+x_n)^k\rangle = \delta_{k\ell}\int_{U(n)} |\operatorname{Tr} M|^{2k} dM. \qquad (1.4.14)
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&\int_{U(n)} (\operatorname{tr}(M+\bar M))^k dM \\
&\quad = \sum_{0\le j\le k} \binom{k}{j} \int_{U(n)} ((\operatorname{tr} M)^j(\overline{\operatorname{tr} M})^{k-j}) dM \\
&\quad = \begin{cases} 0, \ \text{if } k \text{ is odd (because then } j\ne k-j \text{ for all } 0\le j\le k) \\ \binom{k}{k/2} \int_{U(n)} |\operatorname{tr} M|^k dM, \ \text{if } k \text{ is even.} \end{cases} \qquad (1.4.15)
\end{aligned}
$$

Combining the three identities (1.4.13), (1.4.14) and (1.4.15) leads to

$$
\#\{\pi_k\in S_k \text{ such that } L(\pi_k)\le n\} = \binom{2k}{k}^{-1} \int_{U(n)} (\operatorname{Tr}(M+\bar M))^{2k} dM. \qquad (1.4.16)
$$

Finally

$$
\begin{aligned}
&\sum_{n=0}^{\infty}\frac{t^n}{n!}P(L(\pi_n)\leq \ell)\\
&\quad = \sum_{n=0}^{\infty}\frac{t^n}{n!}\frac{\#\{\pi_n\in S_n \bigm| L(\pi_n)\leq \ell\}}{n!}\\
&\quad = \sum_{n=0}^{\infty}\frac{t^n}{(n!)^2}\begin{pmatrix}2n\\ n\end{pmatrix}^{-1}\int_{U(\ell)}(\mathrm{tr}(M+\bar{M}))^{2n}dM\\
&\quad = \sum_{n=0}^{\infty}\frac{(\sqrt{t})^{2n}}{(2n)!}\int_{U(\ell)}(\mathrm{tr}(M+\bar{M}))^{2n}dM\\
&\quad = \int_{U(\ell)}e^{\sqrt{t}\,\mathrm{Tr}(M+\bar{M})}dM\\
&\quad = \frac{1}{\ell!}\int_{[0,2\pi]^{\ell}}e^{\sqrt{t}(z_1+z_1^{-1}+\ldots+z_\ell+z_\ell^{-1})}\prod_{1\leq j<k\leq \ell}|e^{i\theta_j}-e^{i\theta_k}|^2\prod_{1\leq k\leq\ell}\frac{d\theta_k}{2\pi},\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{where } z_k=e^{i\theta_k},\\
&\quad = \frac{1}{\ell!}\int_{[0,2\pi]^{\ell}}\prod_{1\leq j<k\leq\ell}|e^{i\theta_j}-e^{i\theta_k}|^2\prod_{k=1}^{\ell}e^{2\sqrt{t}\cos\theta_k}\frac{d\theta_k}{2\pi},
\end{aligned}
$$

showing (1.4.4) of Proposition 1.1. The latter also equals:

$$
\begin{aligned}
&= \frac{1}{\ell!}\int_{(S^1)^{\ell}}\Delta_\ell(z)\Delta_\ell(\bar{z})\prod_{k=1}^{\ell}\left(e^{\sqrt{t}(z_k+\bar{z}_k)}\frac{dz_k}{2\pi i z_k}\right)\\
&= \frac{1}{\ell!}\int_{(S^1)^{\ell}}\sum_{\sigma\in S_\ell}\det\left(z_{\sigma(m)}^{k-1}\bar{z}_{\sigma(m)}^{m-1}\right)_{1\leq k,m\leq\ell}\prod_{k=1}^{\ell}\left(e^{\sqrt{t}(z_k+\bar{z}_k)}\frac{dz_k}{2\pi i z_k}\right)\\
&= \frac{1}{\ell!}\sum_{\sigma\in S_\ell}\det\left(\int_{S^1}z_k^{k-1}\bar{z}_k^{m-1}e^{\sqrt{t}(z_k+\bar{z}_k)}\frac{dz_k}{2\pi i z_k}\right)_{1\leq k,m\leq\ell}\\
&= \det\left(\int_0^{2\pi}e^{2\sqrt{t}\cos\theta}e^{i(k-m)\theta}d\theta\right)_{1\leq k,m\leq\ell},
\end{aligned}
$$

confirming Gessel's result (1.4.3).

The proof of the second relation (1.4.5) of Proposition 1.1 is based on the following computation:

$$\begin{aligned}
\int_{O(n)} (\operatorname{Tr} M)^k dM &= \sum_{\substack{|\lambda|=k \\ \hat\lambda_1 \le n}} f^\lambda \int_{O(n)} s_\lambda(M) dM, \quad \text{using (1.4.10)}, \\
&= \sum_{\substack{|\lambda|=k \\ \hat\lambda_1 \le n \\ \lambda_i \text{ even}}} f^\lambda \ , \quad \text{using Lemma 1.2}, \\
&= \sum_{\substack{|\lambda|=k \\ \lambda_1 \le n \\ \hat\lambda_i \text{ even}}} f^\lambda \ , \quad \text{using duality}, \\
&= \# \left\{ \begin{array}{l} (P,P), P \text{ standard Young tableau of shape } \lambda \\ \text{with } |\lambda| = k,\ \lambda_1 \le n,\ \hat\lambda_i \text{ even} \end{array} \right\} \\
&= \# \left\{ \pi_k^0 \in S_k^0, \text{ no fixed points and } L(\pi_k^0) \le n \right\}.
\end{aligned} \tag{1.4.17}$$

In the last equality, we have used property (1.4.11): an involution has no fixed points iff all columns of P have even length. Since all columns $\hat\lambda_i$ have even length, it follows that $|\lambda| = k$ is even and then only is $\int_{O(n)} (\operatorname{Tr} M)^k dM > 0$; otherwise $= 0$. Finally, one computes,

$$\begin{aligned}
&\sum_{k=0}^\infty \frac{(t^2/2)^k}{k!} P\left(L(\pi_{2k}^0) \le n,\ \pi_{2k}^0 \in S_{2k}^0 \right) \\
&\quad= \sum_{k=0}^\infty \frac{t^{2k}}{2^k k!} \frac{2^k k!}{(2k)!} \# \{ \pi_{2k}^0 \in S_{2k}^0,\ L(\pi_{2k}^0) \le n \} \ , \quad \text{using (1.4.1)}, \\
&\quad= \sum_{k=0}^\infty \frac{t^k}{k!} \# \{ \pi_k^0 \in S_k^0,\ L(\pi_k^0) \le n \} \\
&\quad= \sum_{k=0}^\infty \frac{t^k}{k!} \int_{O(n)} (\operatorname{Tr} M)^k dM \ , \quad \text{using (1.4.17)}, \\
&\quad= \int_{O(n)} e^{t \operatorname{Tr} M} dM,
\end{aligned}$$

ending the proof of Proposition 1.1. ∎

2 Integrals, vertex operators and Virasoro relations

In section 1, we discussed random matrix problems over different finite and infinite matrix ensembles, generating functions for the statistics of the length of longest increasing sequences in random permutations and involutions. One can also consider two Hermitian random matrix ensembles, coupled together. All those problems lead to matrix integrals or Fredholm determinants, which we list here: ($\beta = 2, 1, 4$)

- $$\int_{\mathcal{H}_n(E),\ \mathcal{S}_n(E)\ \text{or}\ \mathcal{T}_n(E)} e^{-TrV(M)} dM = c_n \int_{E^n} |\Delta_n(z)|^\beta \prod_1^n \rho(z_k) dz_k$$
- $$\int\int_{\mathcal{H}_n^2(E_1 \times E_2)} dM_1 dM_2\, e^{-\frac{1}{2}\operatorname{Tr}(M_1^2 + M_2^2 - 2cM_1M_2)}$$
- $$\int_{O(n)} e^{x \operatorname{Tr} M} dM$$
- $$\int_{U(n)} e^{\sqrt{x} \operatorname{Tr}(M + \bar{M})} dM$$
- $\det\big(I - \lambda K(y,z) I_E(z)\big)$ with $K(y,z)$ as in (1.2.4). (2.0.1)

The point is that each of these quantities admit a natural deformation, by inserting time variables $t_1, t_2, \ldots$ and possibly a second set $s_1, s_2, \ldots$, in a seemingly *ad hoc* way. Each of these integrals or Fredholm determinant is then a fixed point for a natural *vertex operator*, which generates a Virasoro-like algebra. These new integrals in $t_1, t_2, \ldots$ are all annihilated by the precise subalgebra of the Virasoro generators, which annihilates τ_0. This will be the topic of this section.

2.1 Beta-integrals

2.1.1 Virasoro constraints for Beta-integrals

Consider weights of the form $\rho(z)dz := e^{-V(z)}dz$ on an interval $F = [A, B] \subseteq \mathbb{R}$, with rational logarithmic derivative and subjected to the following boundary conditions:

$$-\frac{\rho'}{\rho} = V' = \frac{g}{f} = \frac{\sum_0^\infty b_i z^i}{\sum_0^\infty a_i z^i}, \quad \lim_{z \to A,B} f(z)\rho(z) z^k = 0 \text{ for all } k \geq 0, \tag{2.1.1}$$

and a disjoint union of intervals,

$$E = \bigcup_1^{2r} [c_{2i-1}, c_{2i}] \subset F \subset \mathbb{R}. \tag{2.1.2}$$

These data define an algebra of differential operators

$$\mathcal{B}_k = \sum_1^{2r} c_i^{k+1} f(c_i) \frac{\partial}{\partial c_i}. \tag{2.1.3}$$

Take the first type of integrals in the list (2.0.1) for general $\beta > 0$, thus generalizing the integrals, appearing in the probabilities (1.1.9),(1.1.11) and (1.1.18). Consider t-deformations of such integrals, for general (fixed) $\beta > 0$: ($t := (t_1, t_2, ...)$, $c = (c_1, c_2, ..., c_{2r})$ and $z = (z_1, ..., z_n)$)

$$I_n(t, c; \beta) := \int_{E^n} |\Delta_n(z)|^\beta \prod_{k=1}^n \left(e^{\sum_1^\infty t_i z_k^i} \rho(z_k) dz_k \right) \text{ for } n > 0. \tag{2.1.4}$$

The main statement of this section is Theorem 2.1, whose proof will be outlined in the next subsection. The central charge (2.1.9) has already appeared in the work of Awata et al. [16].

Theorem 2.1 (Adler-van Moerbeke [3, 6]) *The multiple integrals*

$$I_n(t, c; \beta) := \int_{E^n} |\Delta_n(z)|^\beta \prod_{k=1}^n \left(e^{\sum_1^\infty t_i z_k^i} \rho(z_k) dz_k \right), \textit{ for } n > 0 \tag{2.1.5}$$

and

$$I_n(t, c; \frac{4}{\beta}) := \int_{E^n} |\Delta_n(z)|^{4/\beta} \prod_{k=1}^n \left(e^{\sum_1^\infty t_i z_k^i} \rho(z_k) dz_k \right), \textit{ for } n > 0, \tag{2.1.6}$$

with $I_0 = 1$, satisfy respectively the following Virasoro constraints[10] *for all $k \geq -1$:*

$$\begin{gathered}\left(-\mathcal{B}_k + \sum_{i \geq 0} \left(a_i \ {}^\beta\mathbb{J}^{(2)}_{k+i,n}(t, n) - b_i \ {}^\beta\mathbb{J}^{(1)}_{k+i+1,n}(t, n) \right) \right) I_n(t, c; \beta) = 0, \\ \left(-\mathcal{B}_k + \sum_{i \geq 0} \left(a_i \ {}^\beta\mathbb{J}^{(2)}_{k+i,n}(-\frac{\beta t}{2}, -\frac{2n}{\beta}) + \frac{\beta b_i}{2} \ {}^\beta\mathbb{J}^{(1)}_{k+i+1,n}(-\frac{\beta t}{2}, -\frac{2n}{\beta}) \right) \right) I_n(t, c; \frac{4}{\beta}) = 0,\end{gathered} \tag{2.1.7}$$

in terms of the coefficients a_i, b_i of the rational function $(-\log \rho)'$ and the end points c_i of the subset E, as in (2.1.1) to (2.1.2). For all $n \in \mathbb{Z}$, the ${}^\beta\mathbb{J}^{(2)}_{k,n}(t, n)$ and ${}^\beta\mathbb{J}^{(1)}_{k,n}(t, n)$ form a Virasoro and a Heisenberg algebra respectively, interacting as follows

$$\begin{aligned} \left[{}^\beta\mathbb{J}^{(2)}_{k,n}, \ {}^\beta\mathbb{J}^{(2)}_{\ell,n} \right] &= (k - \ell) \ {}^\beta\mathbb{J}^{(2)}_{k+\ell,n} + c \left(\frac{k^3 - k}{12} \right) \delta_{k,-\ell} \\ \left[{}^\beta\mathbb{J}^{(2)}_{k,n}, \ {}^\beta\mathbb{J}^{(1)}_{\ell,n} \right] &= -\ell \ \ {}^\beta\mathbb{J}^{(1)}_{k+\ell,n} + c' k(k+1) \delta_{k,-\ell}. \\ \left[{}^\beta\mathbb{J}^{(1)}_{k,n}, \ {}^\beta\mathbb{J}^{(1)}_{\ell,n} \right] &= \frac{k}{\beta} \delta_{k,-\ell}, \end{aligned} \tag{2.1.8}$$

[10] When E equals the whole range F, then the $\mathcal{B}_k$'s are absent in the formulae (2.1.7).

with central charge

$$c = 1 - 6\left(\left(\frac{\beta}{2}\right)^{1/2} - \left(\frac{\beta}{2}\right)^{-1/2}\right)^2 \quad \textit{and} \quad c' = \left(\frac{1}{\beta} - \frac{1}{2}\right). \tag{2.1.9}$$

<u>*Remark 1:*</u> The ${}^{\beta}\mathbb{J}^{(2)}_{k,n}$'s are defined as follows:

$${}^{\beta}\mathbb{J}^{(2)}_{k,n} = \frac{\beta}{2}\sum_{i+j=k} :\ {}^{\beta}\mathbb{J}^{(1)}_{i,n}\ {}^{\beta}\mathbb{J}^{(1)}_{j,n} : + \left(1 - \frac{\beta}{2}\right)\left((k+1)\ {}^{\beta}\mathbb{J}^{(1)}_{k,n} - k\mathbb{J}^{(0)}_{k,n}\right). \tag{2.1.10}$$

Componentwise, we have

$${}^{\beta}\mathbb{J}^{(1)}_{k,n}(t,n) = {}^{\beta}J^{(1)}_k + nJ^{(0)}_k \text{ and } {}^{\beta}\mathbb{J}^{(0)}_{k,n} = nJ^{(0)}_k = n\delta_{0k}$$

and

$$\begin{aligned} &{}^{\beta}\mathbb{J}^{(2)}_{k,n}(t,n) \\ &= \left(\frac{\beta}{2}\right)\ {}^{\beta}J^{(2)}_k + \left(n\beta + (k+1)(1-\frac{\beta}{2})\right)\ {}^{\beta}J^{(1)}_k + n\Big((n-1)\frac{\beta}{2}+1\Big)J^{(0)}_k, \end{aligned}$$

where

$$\begin{aligned} {}^{\beta}J^{(1)}_k &= \frac{\partial}{\partial t_k} + \frac{1}{\beta}(-k)t_{-k} \\ {}^{\beta}J^{(2)}_k &= \sum_{i+j=k}\frac{\partial^2}{\partial t_i \partial t_j} + \frac{2}{\beta}\sum_{-i+j=k} it_i\frac{\partial}{\partial t_j} + \frac{1}{\beta^2}\sum_{-i-j=k} it_i jt_j. \end{aligned} \tag{2.1.11}$$

We put n explicitly in ${}^{\beta}\mathbb{J}^{(2)}_{\ell,n}(t,n)$ to indicate the nth component contains n explicitly, besides t. Note that for $\beta = 2$, (2.1.10) becomes particularly simple:

$${}^{\beta}\mathbb{J}^{(2)}_{k,n}\Big|_{\beta=2} = \sum_{i+j=k} :\ {}^{2}\mathbb{J}^{(1)}_{i,n}\ {}^{2}\mathbb{J}^{(1)}_{j,n} : .$$

<u>*Remark 2:*</u> The Heisenberg and Virasoro generators satisfy the following *duality* properties:

$$\begin{aligned} {}^{\frac{4}{\beta}}\mathbb{J}^{(2)}_{\ell,n}\Big(t,n\Big) &= {}^{\beta}\mathbb{J}^{(2)}_{\ell,n}\left(-\frac{\beta t}{2}, -\frac{2n}{\beta}\right),\ n \in \mathbb{Z} \\ {}^{\frac{4}{\beta}}\mathbb{J}^{(1)}_{\ell,n}\Big(t,n\Big) &= -\frac{\beta}{2}\ {}^{\beta}\mathbb{J}^{(1)}_{\ell,n}\left(-\frac{\beta t}{2}, -\frac{2n}{\beta}\right),\ n > 0. \end{aligned} \tag{2.1.12}$$

In (2.1.7), ${}^{\beta}\mathbb{J}^{(2)}_{\ell,n}(-\beta t/2, -2n/\beta)$ means that the variable n, which appears in the nth component, gets replaced by $2n/\beta$ and t by $-\beta t/2$.

<u>*Remark 3:*</u> Theorem 2.1 states that the integrals (2.1.5) and (2.1.6) satisfy two sets of differential equations (2.1.7) respectively. Of course, the second integral also satisfies the first set of equations, with β replaced by $4/\beta$.

2.1.2 Proof: β-integrals as fixed points of vertex operators

Theorem 2.1 can be established by using the invariance of the integral under the transformation $z_i \mapsto z_i + \varepsilon f(z_i) z_i^{k+1}$ of the integration variables. However, the most transparent way to prove Theorem 2.1 is via vector vertex operators, for which the β-integrals are fixed points. This is a technique which has been used by us, already in [2]. Indeed, define the (vector) vertex operator, for $t=(t_1,t_2,...)\in\mathbb{C}^\infty,\ u\in\mathbb{C}$, and setting $\chi(z) := (1,z,z^2,...)$.

$$\mathbb{X}_\beta(t,u) = \Lambda^{-1} e^{\sum_1^\infty t_i u^i} e^{-\beta\sum_1^\infty \frac{u^{-i}}{i}\frac{\partial}{\partial t_i}} \chi(|u|^\beta), \tag{2.1.13}$$

which acts on vectors $f(t)=(f_0(t),f_1(t),...)$ of functions, as follows[11]

$$\left(\mathbb{X}_\beta(t,u)f(t)\right)_n = e^{\sum_1^\infty t_i u^i}\left(|u|^\beta\right)^{n-1} f_{n-1}(t-\beta[u^{-1}]).$$

For the sake of these arguments, it is convenient to introduce the following vector Virasoro generators: ${}^\beta\mathbb{J}_k^{(i)}(t) := (\ {}^\beta\mathbb{J}_{k,n}^{(i)}(t,n))_{n\in\mathbb{Z}}$.

Proposition 2.2 *The multiplication operator z^k and the differential operators $\frac{\partial}{\partial z}z^{k+1}$ with $z\in\mathbb{C}^*$, acting on the vertex operator $\mathbb{X}_\beta(t,z)$, have realizations as commutators, in terms of the Heisenberg and Virasoro generators ${}^\beta\mathbb{J}_k^{(1)}(t,n)$ and ${}^\beta\mathbb{J}_k^{(2)}(t,n)$:*

$$\begin{aligned} z^k\mathbb{X}_\beta(t,z) &= \left[\ {}^\beta\mathbb{J}_k^{(1)}(t), \mathbb{X}_\beta(t,z)\right] \\ \frac{\partial}{\partial z}z^{k+1}\mathbb{X}_\beta(t,z) &= \left[\ {}^\beta\mathbb{J}_k^{(2)}(t), \mathbb{X}_\beta(t,z)\right]. \end{aligned} \tag{2.1.14}$$

Corollary 2.3 *Given a weight $\rho(z)dz$ on $\mathbb{R}$ satisfying (2.1.1), we have*

$$\frac{\partial}{\partial z}z^{k+1}f(z)\mathbb{X}_\beta(t,z)\rho(z) = \left[\sum_{i\geq 0}\left(a_i\ {}^\beta\mathbb{J}_{k+i}^{(2)}(t) - b_i\ {}^\beta\mathbb{J}_{k+i+1}^{(1)}(t)\right), \mathbb{X}_\beta(t,z)\rho(z)\right]. \tag{2.1.15}$$

Proof: Using (2.1.13) in the last line, compute

$$\begin{aligned} &\frac{\partial}{\partial z}z^{k+1}f(z)\mathbb{X}_\beta(t,z)\rho(z) \\ &\quad= \left(\frac{\rho'(z)}{\rho(z)}f(z)\right)z^{k+1}\mathbb{X}_\beta(t,z)\rho(z) + \rho(z)\frac{\partial}{\partial z}\left(z^{k+1}f(z)\mathbb{X}_\beta(t,z)\right) \\ &\quad= -\left(\sum_0^\infty b_i z^{k+i+1}\mathbb{X}_\beta(t,z)\right)\rho(z) + \rho(z)\frac{\partial}{\partial z}\left(\sum_0^\infty a_i z^{k+i+1}\mathbb{X}_\beta(t,z)\right) \\ &\quad= -\left[\sum_0^\infty b_i\ {}^\beta\mathbb{J}_{k+i+1}^{(1)}, \mathbb{X}_\beta(t,z)\rho(z)\right] + \left[\sum_0^\infty a_i\ {}^\beta\mathbb{J}_{k+i}^{(2)}, \mathbb{X}_\beta(t,z)\rho(z)\right] \end{aligned} \tag{2.1.16}$$

[11] For $\alpha\in\mathbb{C}$, define $[\alpha] := (\alpha, \frac{\alpha^2}{2}, \frac{\alpha^3}{3}, ...)\in\mathbb{C}^\infty$. The operator Λ is the shift matrix, with zeroes everywhere, except for 1's just above the diagonal, i.e., $(\Lambda v)_n = v_{n+1}$.

establishing (2.1.14). ■

Giving the weight $\rho_E(u)du = \rho(u)I_E(u)du$, with ρ and E as before, define the integrated vector vertex operator

$$\mathbb{Y}_\beta(t,\rho_E) := \int_E du\rho(u)\mathbb{X}_\beta(t,u). \tag{2.1.17}$$

and the vector operator

$$\begin{aligned} \mathcal{D}_k &:= \mathcal{B}_k \quad - \quad \mathcal{V}_k \\ &:= \sum_1^{2r} c_i^{k+1} f(c_i)\frac{\partial}{\partial c_i} - \sum_{i\geq 0}\left(a_i\ {}^\beta\mathbb{J}^{(2)}_{k+i}(t) - b_i\ {}^\beta\mathbb{J}^{(1)}_{k+i+1}(t)\right), \end{aligned} \tag{2.1.18}$$

consisting of a c-dependent boundary part $\mathcal{B}_k$ and a (t,n)-dependent Virasoro part $\mathcal{V}_k$.

Proposition 2.4 *The following commutation relation holds:*

$$[\mathcal{D}_k, \mathbb{Y}_\beta(t,\rho_E)] = 0 \tag{2.1.19}$$

Proof: Integrating both sides of (2.1.14) over E, one computes:

$$\begin{aligned} \int_E dz\frac{\partial}{\partial z}\left(z^{k+1}f(z)\mathbb{X}_\beta(t,z)\rho(z)\right) &= \sum_1^{2r}(-1)^i c_i^{k+1} f(c_i)\mathbb{X}_\beta(t,c_i)\rho(c_i) \\ &= \sum_1^{2r} c_i^{k+1} f(c_i)\frac{\partial}{\partial c_i}\int_E \mathbb{X}_\beta(t,z)\rho(z)dz \\ &= \left[\mathcal{B}_k, \mathbb{Y}_\beta(t,\rho_E)\right], \end{aligned} \tag{2.1.20}$$

while on the other hand

$$\begin{aligned} &\int_E dz\left[\sum_{i\geq 0}\left(a_i\ {}^\beta\mathbb{J}^{(2)}_{k+i} - b_i\ {}^\beta\mathbb{J}^{(1)}_{k+i+1}\right), \mathbb{X}_\beta(t,z)\rho(z)\right] \\ &\quad = \left[\sum_{i\geq 0}\left(a_i\ {}^\beta\mathbb{J}^{(2)}_{k+i} - b_i\ {}^\beta\mathbb{J}^{(1)}_{k+i+1}\right), \int_{\mathbb{R}} dz\rho_E(z)\mathbb{X}_\beta(t,z)\right] \\ &\quad = \left[\mathcal{V}_k, \mathbb{Y}_\beta(t,\rho_E)\right]. \end{aligned} \tag{2.1.21}$$

Subtracting both expressions (2.1.19) and (2.1.20) yields

$$0 = [\mathcal{B}_k - \mathcal{V}_k, \mathbb{Y}_\beta(t,\rho_E)] = \left[\mathcal{D}_k, \mathbb{Y}_\beta(t,\rho_E)\right],$$

concluding the proof of proposition 2.4. ■

Proposition 2.5 *The column vector,*

$$I(t) := \left(\int_{E^n} |\Delta_n(z)|^\beta \prod_{k=1}^n e^{\sum_1^\infty t_i z_k^i} \rho(z_k) dz_k \right)_{n \geq 0}$$

is a fixed point for the vertex operator $\mathbb{Y}_\beta(t,\rho_E)$: (see definition (2.1.17))

$$\left(\mathbb{Y}_\beta(t,\rho_E) I\right)_n = I_n, \ \ n \geq 1. \tag{2.1.22}$$

Proof: We have

$$\begin{aligned} I_n(t) &= \int_{\mathbb{R}^n} |\Delta_n(z)|^\beta \prod_{k=1}^n \left(e^{\sum_1^\infty t_i z_k^i} I_E(z_k) \rho(z_k) dz_k \right) \\ &= \int_{\mathbb{R}} du \rho_E(u) e^{\sum_1^\infty t_i u^i} |u|^{\beta(n-1)} \\ &\qquad \int_{\mathbb{R}^{n-1}} \prod_{k=1}^{n-1} \left| 1 - \frac{z_k}{u} \right|^\beta |\Delta_{n-1}(z)|^\beta \prod_{k=1}^{n-1} \left(e^{\sum_1^\infty t_i z_k^i} \rho_E(z_k) dz_k \right) \\ &= \int_{\mathbb{R}} du \rho_E(u) e^{\sum_1^\infty t_i u^i} |u|^{\beta(n-1)} \\ &\qquad e^{-\beta \sum_1^\infty \frac{u^{-i}}{i} \frac{\partial}{\partial t_i}} \int_{\mathbb{R}^{n-1}} |\Delta_{n-1}(z)|^\beta \prod_{k=1}^{n-1} \left(e^{\sum_1^\infty t_i z_k^i} \rho_E(z_k) dz_k \right) \\ &= \int_{\mathbb{R}} du \rho_E(u) |u|^{\beta(n-1)} e^{\sum_1^\infty t_i u^i} e^{-\beta \sum_1^\infty \frac{u^{-i}}{i} \frac{\partial}{\partial t_i}} \tau_{n-1}(t) \\ &= \left(\mathbb{Y}_\beta(t,\rho_E) I(t) \right)_n. \end{aligned} \tag{2.1.23}$$

■

<u>*Proof of Theorem 2.1*</u>: From proposition 2.4 it follows that for $n \geq 1$,

$$\begin{aligned} 0 &= \left[\mathcal{D}_k, (\mathbb{Y}_\beta(t,\rho_E))^n\right] I \\ &= \mathcal{D}_k \mathbb{Y}_\beta(t,\rho_E)^n I - \mathbb{Y}_\beta(t,\rho_E)^n \mathcal{D}_k I. \end{aligned} \tag{2.1.24}$$

Taking the nth component for $n \geq 1$ and $k \geq -1$, and setting $X_\beta(t,u) = e^{\sum t_i u^i} e^{-\beta \sum \frac{u^{-i}}{i} \frac{\partial}{\partial t_i}}$, and using (2.1.21)

$$\begin{aligned} 0 &= \left(\mathcal{D}_k I - \mathbb{Y}_\beta(t,\rho_E)^n \mathcal{D}_k I\right)_n \\ &= (\mathcal{D}_k I)_n - \int du \rho_E(u) X_\beta(t;u) (|u|^\beta)^{n-1} \ldots \int du \rho_E(u) X_\beta(t;u) (\mathcal{D}_k I)_0 \\ &= (\mathcal{D}_k I)_n. \end{aligned}$$

Indeed $(\mathcal{D}_k I)_0 = 0$ for $k \geq -1$, since $\tau_0 = 1$ and $\mathcal{D}_k$ involves ${}^{\beta}J_k^{(2)}$, ${}^{\beta}J_k^{(1)}$ and $J_k^{(0)}$ for $k \geq -1$:

$$\begin{cases} {}^{\beta}J_k^{(2)} \text{ is pure differentiation for } k \geq -1; \\ {}^{\beta}J_k^{(1)} \text{is pure differentiation, except for } k = -1; \text{ but} \\ \ {}^{\beta}J_{-1}^{(1)} \text{ appears with coefficient } n\beta, \text{ which vanishes for } n = 0; \\ J_k^{(0)} \text{ appears with coefficient } n((n-1)\frac{\beta}{2}+1), \text{ vanishing for } n = 0. \end{cases}$$

■

2.1.3 Examples

Example 1 : Gaussian β-integrals

The weight and the a_i and b_i, as in (2.1.1), are given by

$$\rho(z) = e^{-V(z)} = e^{-z^2}, \quad V' = g/f = 2z,$$

$$a_0 = 1, b_0 = 0, b_1 = 2, \text{ and all other } a_i, b_i = 0.$$

From (2.1.5), the integrals

$$I_n = \int_{E^n} |\Delta_n(z)|^{\beta} \prod_{k=1}^{n} e^{-z_k^2 + \sum_{i=1}^{\infty} t_i z_k^i} dz_k \tag{2.1.25}$$

satisfy the Virasoro constraints, for $k \geq -1$,

$$-\mathcal{B}_k I_n = -\sum_1^{2r} c_i^{k+2} \frac{\partial}{\partial c_i} I_n = \left(-\ {}^{\beta}\mathbb{J}_{k+1,n}^{(2)} - a\ {}^{\beta}\mathbb{J}_{k+1,n}^{(1)} +\ {}^{\beta}\mathbb{J}_{k+2,n}^{(1)}\right) I_n. \tag{2.1.26}$$

Introducing the following notation

$$\sigma_i = (n - \frac{i+1}{2})\beta + i + 1 - b_0 = (n - \frac{i+1}{2})\beta + i + 1, \tag{2.1.27}$$

the first three constraints have the following form, upon setting $F_n = \log I_n$,

$$\begin{aligned} -\mathcal{B}_{-1}F &= \left(2\frac{\partial}{\partial t_1} - \sum_{i\geq 2} it_i \frac{\partial}{\partial t_{i-1}}\right) F - nt_1\ , \ -\mathcal{B}_0 F = \left(2\frac{\partial}{\partial t_2} - \sum_{i\geq 1} it_i \frac{\partial}{\partial t_i}\right) F - \frac{n}{2}\sigma_1 \\ -\mathcal{B}_1 F &= \left(2\frac{\partial}{\partial t_3} - \sigma_1 \frac{\partial}{\partial t_1} - \sum_{i\geq 1} it_i \frac{\partial}{\partial t_{i+1}}\right) F. \end{aligned} \tag{2.1.28}$$

For later use, take the linear combinations

$$\mathcal{D}_1 = -\frac{1}{2}\mathcal{B}_{-1}, \quad \mathcal{D}_2 = -\frac{1}{2}\mathcal{B}_0, \quad \mathcal{D}_3 = -\frac{1}{2}\left(\mathcal{B}_1 + \frac{\sigma_1}{2}\mathcal{B}_{-1}\right), \tag{2.1.29}$$

such that each $\mathcal{D}_i$ contains the pure term $\partial F/\partial t_i$, i.e., $\mathcal{D}_i F = \frac{\partial F}{\partial t_i} + ...$

Example 2 (Laguerre β-integrals)

Here, the weight and the a_i and b_i, as in (2.1.1), are given by

$$e^{-V}=z^a e^{-z},\quad V'=\frac{g}{f}=\frac{z-a}{z},$$

$$a_0=0,\ a_1=1,\ b_0=-a,\ b_1=1,\ \text{and all other } a_i,b_i=0.$$

Thus from (2.1.4), the integrals

$$I_n=\int_{E^n}|\Delta_n(z)|^\beta\prod_{k=1}^n z_k^a e^{-z_k+\sum_{i=1}^\infty t_i z_k^i}dz_k \tag{2.1.30}$$

satisfy the Virasoro constraints, for $k\geq -1$,

$$-\mathcal{B}_k I_n=-\sum_1^{2r}c_i^{k+2}\frac{\partial}{\partial c_i}I_n=\Big(-\ {}^\beta\mathbb{J}^{(2)}_{k+1,n}-a\ {}^\beta\mathbb{J}^{(1)}_{k+1,n}+\ {}^\beta\mathbb{J}^{(1)}_{k+2,n}\Big)I_n. \tag{2.1.31}$$

Introducing the following notation

$$\sigma_i\ =\ =(n-\frac{i+1}{2})\beta+i+1-b_0=(n-\frac{i+1}{2})\beta+i+1+a,$$

the first three have the form, upon setting $F=F_n=\log I_n$,

$$\begin{aligned}
-\mathcal{B}_{-1}F &= \left(\frac{\partial}{\partial t_1}-\sum_{i\geq1}it_i\frac{\partial}{\partial t_i}\right)F-\frac{n}{2}(\sigma_1+a)\\
-\mathcal{B}_0F &= \left(\frac{\partial}{\partial t_2}-\sigma_1\frac{\partial}{\partial t_1}-\sum_{i\geq1}it_i\frac{\partial}{\partial t_{i+1}}\right)F\\
-\mathcal{B}_1F &= \left(\frac{\partial}{\partial t_3}-\sigma_2\frac{\partial}{\partial t_2}-\sum_{i\geq1}it_i\frac{\partial}{\partial t_{i+2}}-\frac{\beta}{2}\frac{\partial^2}{\partial t_1^2}\right)F-\frac{\beta}{2}\left(\frac{\partial F}{\partial t_1}\right)^2.
\end{aligned}$$

Again, replace the operators $\mathcal{B}_i$ by linear combinations $\mathcal{D}_i$, such that $\mathcal{D}_iF=\frac{\partial F}{\partial t_i}+...$,

$$\mathcal{D}_1=-\mathcal{B}_{-1},\quad \mathcal{D}_2=-\mathcal{B}_0-\sigma_1\mathcal{B}_{-1},\quad \mathcal{D}_3=-\mathcal{B}_1-\sigma_2\mathcal{B}_0-\sigma_1\sigma_2\mathcal{B}_{-1}. \tag{2.1.32}$$

Example 3 (Jacobi β-integral)

This case is particularly important, because it covers not only the first integral, but also the second integral in the list (2.0.1), used in the problem of random permutations. The weight and the a_i and b_i, as in (2.1.1), are given by

$$\rho(z):=e^{-V}=(1-z)^a(1+z)^b,V'=\frac{g}{f}=\frac{a-b+(a+b)z}{1-z^2}$$

$$a_0=1, a_1=0, a_2=-1, b_0=a-b, b_1=a+b, \text{ and all other } a_i, b_i=0.$$

The integrals

$$I_n=\int_{E^n}|\Delta_n(z)|^{\beta}\prod_{k=1}^{n}(1-z_k)^a(1+z_k)^b e^{\sum_{i=1}^{\infty}t_iz_k^i}dz_k \tag{2.1.33}$$

satisfy the Virasoro constraints ($k\geq -1$):

$$\begin{aligned} -\mathcal{B}_kI_n &= -\sum_{1}^{2r}c_i^{k+1}(1-c_i^2)\frac{\partial}{\partial c_i}I_n \\ &= \left({}^{\beta}\mathbb{J}^{(2)}_{k+2,n}-{}^{\beta}\mathbb{J}^{(2)}_{k,n}+b_0\,{}^{\beta}\mathbb{J}^{(1)}_{k+1,n}+b_1\,{}^{\beta}\mathbb{J}^{(1)}_{k+2,n}\right)I_n. \end{aligned} \tag{2.1.34}$$

Introducing $\sigma_i=(n-\frac{i+1}{2})\beta+i+1+b_1$, the first four have the form ($k=-1,0,1,2$)

$$\begin{aligned} -\mathcal{B}_{-1}F &= \left(\sigma_1\frac{\partial}{\partial t_1}+\sum_{i\geq 1}it_i\frac{\partial}{\partial t_{i+1}}-\sum_{i\geq 2}it_i\frac{\partial}{\partial t_{i-1}}\right)F+n(b_0-t_1) \\ -\mathcal{B}_0F &= \left(\sigma_2\frac{\partial}{\partial t_2}+b_0\frac{\partial}{\partial t_1}+\sum_{i\geq 1}it_i(\frac{\partial}{\partial t_{i+2}}-\frac{\partial}{\partial t_i})+\frac{\beta}{2}\frac{\partial^2}{\partial t_1^2}\right)F+\frac{\beta}{2}\left(\frac{\partial F}{\partial t_1}\right)^2-\frac{n}{2}(\sigma_1-b_1) \\ -\mathcal{B}_1F &= \left(\sigma_3\frac{\partial}{\partial t_3}+b_0\frac{\partial}{\partial t_2}-(\sigma_1-b_1)\frac{\partial}{\partial t_1}+\sum_{i\geq 1}it_i(\frac{\partial}{\partial t_{i+3}}-\frac{\partial}{\partial t_{i+1}})+\beta\frac{\partial^2}{\partial t_1\partial t_2}\right)F \\ &\qquad +\beta\frac{\partial F}{\partial t_1}\frac{\partial F}{\partial t_2} \\ -\mathcal{B}_2F &= \left(\sigma_4\frac{\partial}{\partial t_4}+b_0\frac{\partial}{\partial t_3}-(\sigma_2-b_1)\frac{\partial}{\partial t_2}+\sum_{i\geq 1}it_i(\frac{\partial}{\partial t_{i+4}}-\frac{\partial}{\partial t_{i+2}})\right. \\ &\quad \left.+\frac{\beta}{2}(\frac{\partial^2}{\partial t_2^2}-\frac{\partial^2}{\partial t_1^2}+2\frac{\partial^2}{\partial t_1\partial t_3})\right)F+\frac{\beta}{2}\left((\frac{\partial F}{\partial t_2})^2-(\frac{\partial F}{\partial t_1})^2+2\frac{\partial F}{\partial t_1}\frac{\partial F}{\partial t_3}\right). \end{aligned} \tag{2.1.35}$$

2.2 Double matrix integrals

Consider now weights of the form

$$\rho(x,y)=e^{\sum_{i,j\geq 1}r_{ij}x^iy^j}\rho(x)\tilde{\rho}(y), \tag{2.2.1}$$

defined on a product of intervals $F_1\times F_2\subset\mathbb{R}^2$, with rational logarithmic derivative

$$-\frac{\rho'}{\rho}=\frac{g}{f}=\frac{\sum_{i\geq 0}b_ix^i}{\sum_{i\geq 0}a_ix^i} \quad\text{and}\quad -\frac{\tilde{\rho}'}{\tilde{\rho}}=\frac{\tilde{g}}{\tilde{f}}=\frac{\sum_{i\geq 0}\tilde{b}_iy^i}{\sum_{i\geq 0}\tilde{a}_iy^i},$$

satisfying

$$\lim_{x\to\partial F} f(x)\rho(x)x^k = \lim_{y\to\partial\tilde F}\tilde f(y)\tilde\rho(y)y^k = 0 \ \text{ for all } k\geq 0. \tag{2.2.2}$$

Consider subsets of the form

$$E = E_1\times E_2 := \cup_{i=1}^r[c_{2i-1},c_{2i}]\times\cup_{i=1}^s[\tilde c_{2i-1},\tilde c_{2i}]\subset F_1\times F_2\subset\mathbb{R}^2. \tag{2.2.3}$$

A natural deformation of the second integral in the list (2.0.1) is given by the following integrals:

$$I_n(t,s,r;E) = \iint_{E^n}\Delta_n(x)\Delta_n(y)\prod_{k=1}^n e^{\sum_{i=1}^\infty(t_ix_k^i-s_iy_k^i)}\rho(x_k,y_k)dx_kdy_k \tag{2.2.4}$$

In the theorem below, $\mathbb{J}_{k,n}^{(i)}$ and $\tilde{\mathbb{J}}_{k,n}^{(i)}$ are vectors of operators, whose components are given by the operators (2.1.10) for $\beta=1$; i.e.,

$$\mathbb{J}_{k,n}^{(i)}(t) = {}^\beta\mathbb{J}_{k,n}^{(i)}(t)\Big|_{\beta=1}, \tilde{\mathbb{J}}_{k,n}^{(i)}(s) := {}^\beta\mathbb{J}_{k,n}^{(i)}(t)\Big|_{\beta=1,\ t\mapsto -s};$$

thus, from (2.1.10) and (2.1.11), one finds:

$$\mathbb{J}_{k,n}^{(2)}(t) = \frac12\left(J_k^{(2)}(t)+(2n+k+1)J_k^{(1)}(t)+n(n+1)J_k^{(0)}\right), \tag{2.2.5}$$

satisfying the Heisenberg and Virasoro relations (2.1.8), with *central charge* $c=-2$ and $c'=1/2$.

The $a_i,\tilde a_i,b_i,\tilde b_i,c_i,\tilde c_i,r_{ij}$ given by (2.2.1), (2.2.2) and (2.2.3) define differential operators:

$$\begin{aligned}
\mathcal{D}_{k,n}: &= \sum_1^{2r} c_i^{k+1}f(c_i)\frac{\partial}{\partial c_i}-\sum_{i\geq 0}\Big(a_i(\mathbb{J}_{k+i,n}^{(2)}+\sum_{m,\ell\geq 1}mr_{m\ell}\frac{\partial}{\partial r_{m+k+i,\ell}})-b_i\mathbb{J}_{k+i+1,n}^{(1)}\Big)\\
\tilde{\mathcal{D}}_{k,n}: &= \sum_1^{2r} \tilde c_i^{k+1}\tilde f(\tilde c_i)\frac{\partial}{\partial\tilde c_i}-\sum_{i\geq 0}\Big(\tilde a_i(\tilde{\mathbb{J}}_{k+i,n}^{(2)}+\sum_{m,\ell\geq 1}\ell r_{m\ell}\frac{\partial}{\partial r_{m,\ell+k+i}})-\tilde b_i\tilde{\mathbb{J}}_{k+i+1,n}^{(1)}\Big).
\end{aligned} \tag{2.2.6}$$

Theorem 2.6 (Adler-van Moerbeke [3, 4]) *Given $\rho(x,y)$ as in (2.2.1), the integrals*

$$I_n(t,s,r;E) := \iint_{E^n}\Delta_n(x)\Delta_n(y)\prod_{k=1}^n e^{\sum_{i=1}^\infty(t_ix_k^i-s_iy_k^i)}\rho(x_k,y_k)dx_kdy_k \tag{2.2.7}$$

satisfy two families of Virasoro equations for $k\geq -1$:

$$\mathcal{D}_{k,n}I_n(t,s,r;E)=0 \quad \textit{and} \quad \tilde{\mathcal{D}}_{k,n}I_n(t,s,r;E)=0. \tag{2.2.8}$$

The proof of this statement is very similar to the one for β-integrals. Namely, define the vector vertex operator,

$$\mathbb{X}_{12}(t,s;u,v) = \Lambda^{-1} e^{\sum_1^\infty (t_i u^i - s_i v^i)} e^{-\sum_1^\infty (\frac{u^{-i}}{i}\frac{\partial}{\partial t_i} - \frac{v^{-i}}{i}\frac{\partial}{\partial s_i})} \chi(uv), \tag{2.2.9}$$

which, as a consequence of Proposition 2.2, interacts with the operators $\mathbb{J}_k^{(i)}(t) = \left(\mathbb{J}_{k,n}^{(i)}(t,n)\right)_{n\in\mathbb{Z}}$, as follows:

$$\begin{aligned} u^k \mathbb{X}_{12}(t,s;u,v) &= \left[\mathbb{J}_k^{(1)}(t), \mathbb{X}_{12}(t,s;u,v)\right] \\ \frac{\partial}{\partial u} u^{k+1} \mathbb{X}_{12}(t,s;u,v) &= \left[\mathbb{J}_k^{(2)}(t), \mathbb{X}_{12}(t,s;u,v)\right]. \end{aligned} \tag{2.2.10}$$

A similar statement can be made, upon replacing the operators u^k and $\frac{\partial}{\partial u}u^{k+1}$ by v^k and $\frac{\partial}{\partial v}v^{k+1}$, and upon using the $\tilde{\mathbb{J}}_k^{(i)}(s)$'s.

Finally, one checks that the integral vertex operator

$$\mathbb{Y}(t,s;\rho_E) := \iint_E dx dy \rho(x,y) \mathbb{X}_{12}(t,s;x,y) \tag{2.2.11}$$

commutes with the two vectors of differential operators $\mathcal{D}_k = (\mathcal{D}_{k,n})_{n\in\mathbb{Z}}$, as in (2.2.6):

$$\left[\mathcal{D}_k, \mathbb{Y}(t,s;\rho_E)\right] = \left[\tilde{\mathcal{D}}_k, \mathbb{Y}(t,s;\rho_E)\right] = 0,$$

and that the vector $I = (I_0 = 1, I_1, ...)$ of integrals (2.2.7) is a fixed point for $\mathbb{Y}(t,s;\rho_E)$,

$$\mathbb{Y}(t,s;\rho_E) I(t,s,r;E) = I(t,s,r;E).$$

Then, as before, the proof of Theorem 2.6 hinges ultimately on the fact that $\mathcal{D}_{k,0}$ annihilates $I_0 = 1$.

2.3 Integrals over the unit circle

We now deal with the fourth type of integral in the list (2.0.1), which we deform, this time, by inserting two sequences of times $t_1, t_2, ...$ and $s_1, s_2, ...$ The following theorem holds:

Theorem 2.7 (Adler-van Moerbeke [7]) *The multiple integrals over the unit circle S^1,*

$$I_n(t,s) = \int_{(S^1)^n} |\Delta_n(z)|^2 \prod_{k=1}^n e^{\sum_1^\infty (t_i z_k^i - s_i z_k^{-i})} \frac{dz_k}{2\pi i z_k}, \quad n > 0, \tag{2.3.1}$$

with $I_0 = 1$, satisfy an SL(2,$\mathbb{Z}$)-algebra of Virasoro constraints:

$$\mathcal{D}^{\theta}_{k,n} I_n(t,s) = 0, \quad \textit{for} \left\{ \begin{array}{l} k=-1,\ \theta = 0 \\ k=0,\ \ \theta \ \ \textit{arbitrary} \\ k=1,\ \theta=1 \end{array} \right\} \textit{only}, \tag{2.3.2}$$

where the operators $\mathcal{D}^{\theta}_{k,n} := \mathcal{D}^{\theta}_{k,n}(t,s,n)$, $\ k \in \mathbb{Z},\ n \geq 0$ *are given by*

$$\mathcal{D}^{\theta}_{k,n} := \mathbb{J}^{(2)}_{k,n}(t,n) - \mathbb{J}^{(2)}_{-k,n}(-s,n) - k\left(\theta \mathbb{J}^{(1)}_{k,n}(t,n) + (1-\theta)\mathbb{J}^{(1)}_{-k,n}(-s,n)\right), \tag{2.3.3}$$

with $\mathbb{J}^{(i)}_{k,n}(t,n) := {}^{\beta}\mathbb{J}^{(i)}_{k,n}(t,n)\Big|_{\beta=1}$*, as in (2.1.11).*

The explicit expressions are

$$\begin{aligned} \mathcal{D}_{-1} I_n &= \left(\sum_{i\geq 1}(i+1)t_{i+1}\frac{\partial}{\partial t_i} - \sum_{i\geq 2}(i-1)s_{i-1}\frac{\partial}{\partial s_i} + n\left(t_1 + \frac{\partial}{\partial s_1}\right)\right) I_n = 0 \\ \mathcal{D}_0 I_n &= \sum_{i\geq 1}\left(it_i\frac{\partial}{\partial t_i} - is_i\frac{\partial}{\partial s_i}\right) I_n = 0 \\ \mathcal{D}_1 I_n &= \left(-\sum_{i\geq 1}(i+1)s_{i+1}\frac{\partial}{\partial s_i} + \sum_{i\geq 2}(i-1)t_{i-1}\frac{\partial}{\partial t_i} + n\left(s_1 + \frac{\partial}{\partial t_1}\right)\right) I_n = 0. \end{aligned} \tag{2.3.4}$$

Here the key vertex operator is a reduction of $\mathbb{X}_{12}(t,s;u,v)$, defined in the previous section (formula (2.2.9)). For all $k \in \mathbb{Z}$, the vector of operators $\mathcal{D}^{\theta}_k(t,s) = \left(\mathcal{D}^{\theta}_{k,n}(t,s,n)\right)_{n\in\mathbb{Z}}$ form a realization of the first order differential operators $\frac{d}{du}u^{k+1}$, using the vertex operator $\mathbb{X}_{12}(t,s;u,u^{-1})$, namely

$$\frac{d}{du}u^{k+1}\frac{\mathbb{X}_{12}(t,s;u,u^{-1})}{u} = \left[\mathcal{D}^{\theta}_k(t,s), \frac{\mathbb{X}_{12}(t,s;u,u^{-1})}{u}\right]. \tag{2.3.5}$$

Indeed,

$$\begin{aligned} & u\frac{d}{du}u^k\mathbb{X}_{12}(t,s;u,u^{-1}) \\ &= \left(\frac{\partial}{\partial u}u^{k+1} - \frac{\partial}{\partial v}v^{1-k} - k\theta u^k - k(1-\theta)v^{-k}\right)\mathbb{X}_{12}(t,s;u,v)\Big|_{v=-u} \\ &= \left[\mathbb{J}^{(2)}_k(t) - \mathbb{J}^{(2)}_{-k}(-s) - k\left(\theta\mathbb{J}^{(1)}_k(t) + (1-\theta)\mathbb{J}^{(1)}_k(-s)\right), \mathbb{X}_{12}(t,s;u,-u)\right] \\ &= \left[\mathcal{D}^{\theta}_k(t,s), \mathbb{X}_{12}(t,s;u,u^{-1}\right]. \end{aligned}$$

The $\mathcal{D}^{\theta}_k := \mathcal{D}^{\theta}_k(t,s)$ satisfy Virasoro relations with central charge $= 0$,

$$\left[\mathcal{D}^{\theta}_k, \mathcal{D}^{\theta}_{\ell}\right] = (k-\ell)\mathcal{D}^{\theta}_{k+\ell}, \tag{2.3.6}$$

and, from (2.3.5) the following commutation relation holds:

$$\left[\mathcal{D}_k^\theta(t,s), \mathbb{Y}(t,s)\right] = 0, \quad \text{with} \quad \mathbb{Y}(t,s) := \int_{S^1} \frac{du}{2\pi i u} \mathbb{X}_{12}(t,s;u,u^{-1}). \tag{2.3.7}$$

The point is that the column vector $I(t,s) = (I_0, I_1, ...)$ of integrals (2.3.1), is a fixed point for $\mathbb{Y}(t,s)$:

$$(\mathbb{Y}(t,s)I)_n = I_n, \ n \geq 1, \tag{2.3.8}$$

which is shown in a way, similar to Proposition 2.5.
Proof of Theorem 2.7: Here again the proof is similar to the one of Theorem 2.1. Taking the nth component and the nth power of $\mathbb{Y}(t,s)$, with $n \geq 1$, and noticing from the explicit formulae (2.3.4) that $\left(\mathcal{D}_k^\theta(t,s)I\right)_0 = 0$, we have, by means of a calculation similar to the proof of Theorem 2.1,

$$\begin{aligned} 0 &= \left([\mathcal{D}_k^\theta, \mathbb{Y}(t,s)^n]I\right)_n \\ &= \left(\mathcal{D}_k^\theta \mathbb{Y}(t,s)^n I - \mathbb{Y}(t,s)^n \mathcal{D}_k^\theta I\right)_n \\ &= \left(\mathcal{D}_k^\theta I - \mathbb{Y}(t,s)^n \mathcal{D}_k^\theta I\right)_n = \left(\mathcal{D}_k^\theta I\right)_n . \end{aligned}$$

■

3 Integrable systems and associated matrix integrals

3.1 Toda lattice and Hermitian matrix integrals

3.1.1 Toda lattice, factorization of symmetric matrices and orthogonal polynomials

Given a weight $\rho(z) = e^{-V(z)}$ defined as in (2.1.1), the inner-product over $E \subseteq \mathbb{R}$,

$$\langle f, g \rangle_t = \int_E f(z)g(z)\rho_t(z)dz, \quad \text{with } \rho_t(z) := e^{\sum t_i z^i}\rho(z), \tag{3.1.1}$$

leads to a moment matrix

$$m_n(t) = (\mu_{ij}(t))_{0 \leq i,j < n} = (\langle z^i, z^j \rangle_t)_{0 \leq i,j < n}, \tag{3.1.2}$$

which is a *Hänkel matrix*[12], thus symmetric. This is tantamount to $\Lambda m_\infty = m_\infty \Lambda^\top$, where Λ denotes the shift matrix; see footnote 11 . As easily seen, the semi-infinite moment matrix m_∞ evolves in t according to the equations

$$\frac{\partial \mu_{ij}}{\partial t_k} = \mu_{i+k,j}, \text{ and thus } \frac{\partial m_\infty}{\partial t_k} = \Lambda^k m_\infty. \qquad \left(\begin{array}{c}\text{commuting}\\ \text{vector fields}\end{array}\right) \tag{3.1.3}$$

[12] Hänkel means μ_{ij} depends on $i+j$ only

Another important ingredient is the factorization of m_∞ into a lower- times an upper-triangular matrix[13]

$$m_\infty(t) = S(t)^{-1} S(t)^{\top -1}, \qquad S(t) = \begin{array}{l}\text{lower triangular with}\\ \text{non-zero diagonal elements.}\end{array}$$

The main ideas of the following theorem can be found in [2, 5]. Remember $c = (c_1, ..., c_{2r})$ denotes the boundary points of the set E. dM refers to properly normalized Haar measure on $\mathcal{H}_n$.

Theorem 3.1 *The determinants of the moment matrices*

$$\begin{aligned}\tau_n(t,c) := \det m_n(t,c) &= \frac{1}{n!}\int_{E^n} \Delta_n^2(z) \prod_{k=1}^{n} \rho_t(z_k) dz_k \\ &= \int_{\mathcal{H}_n(E)} e^{tr(-V(M)+\sum_1^\infty t_i M^i)} dM, \end{aligned} \tag{3.1.4}$$

satisfy

(i) *Virasoro constraints (2.1.7) for $\beta = 2$,*

$$\left(-\sum_1^{2r} c_i^{k+1} f(c_i) \frac{\partial}{\partial c_i} + \sum_{i \geq 0} \left(a_i \ \mathbb{J}^{(2)}_{k+i,n} - b_i \ \mathbb{J}^{(1)}_{k+i+1,n} \right) \right) \tau_n(t,c) = 0. \tag{3.1.5}$$

(ii) *The KP-hierarchy[14] $(k = 0, 1, 2, \ldots)$*

$$\left(\mathbf{s}_{k+4}\left(\frac{\partial}{\partial t_1}, \frac{1}{2}\frac{\partial}{\partial t_2}, \frac{1}{3}\frac{\partial}{\partial t_3}, \ldots \right) - \frac{1}{2} \frac{\partial^2}{\partial t_1 \partial t_{k+3}} \right) \tau_n \circ \tau_n = 0,$$

of which the first equation reads:

$$\left(\left(\frac{\partial}{\partial t_1} \right)^4 + 3 \left(\frac{\partial}{\partial t_2} \right)^2 - 4 \frac{\partial^2}{\partial t_1 \partial t_3} \right) \log \tau_n + 6 \left(\frac{\partial^2}{\partial t_1^2} \log \tau_n \right)^2 = 0. \tag{3.1.6}$$

(iii) *The standard Toda lattice, i.e., the symmetric tridiagonal matrix*

$$L(t) := S(t) \Lambda S(t)^{-1} = \begin{pmatrix} \frac{\partial}{\partial t_1} \log \frac{\tau_1}{\tau_0} & \left(\frac{\tau_0 \tau_2}{\tau_1^2} \right)^{1/2} & 0 & \\ \left(\frac{\tau_0 \tau_2}{\tau_1^2} \right)^{1/2} & \frac{\partial}{\partial t_1} \log \frac{\tau_2}{\tau_1} & \left(\frac{\tau_1 \tau_3}{\tau_2^2} \right)^{1/2} & \\ 0 & \left(\frac{\tau_1 \tau_3}{\tau_2^2} \right)^{1/2} & \frac{\partial}{\partial t_1} \log \frac{\tau_3}{\tau_2} & \\ & & & \ddots \end{pmatrix} \tag{3.1.7}$$

[13]This factorization is possible for those t's for which $\tau_n(t) := \det m_n(t) \neq 0$ for all $n > 0$.

[14]Given a polynomial $p(t_1, t_2, ...)$, define the customary Hirota symbol $p(\partial_t) f \circ g := p(\frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}, ...) f(t+y) g(t-y)\Big|_{y=0}$. The $\mathbf{s}_\ell$'s are the elementary Schur polynomials $e^{\sum_1^\infty t_i z^i} := \sum_{i \geq 0} \mathbf{s}_i(t) z^i$ and $\mathbf{s}_\ell(\tilde{\partial}) := \mathbf{s}_\ell(\frac{\partial}{\partial t_1}, \frac{1}{2}\frac{\partial}{\partial t_2}, \ldots)$.

satisfies the commuting equations[15]

$$\frac{\partial L}{\partial t_k} = \left[\frac{1}{2}(L^k)_{\mathfrak{s}}, L\right]. \tag{3.1.8}$$

(iv) *Eigenvectors of L: The tridiagonal matrix L admits two independent eigenvectors:*

- $p(t;z) = (p_n(t;z))_{n\geq 0}$ *satisfying* $(L(t)p(t;z))_n = zp_n(t;z)$, $n \geq 0$, *where* $p_n(t;z)$ *are nth degree polynomials in z, depending on* $t \in \mathbb{C}^\infty$, *orthonormal with respect to the t-dependent inner product*[16] *(3.1.1)*

$$\langle p_k(t;z), p_\ell(t;z)\rangle_t = \delta_{k\ell};$$

they are eigenvectors of L, i.e., $L(t)p(t;z) = zp(t;z)$, *and enjoy the following representations:* $(\chi(z) = (1, z, z^2, ...)^\top)$

$$\begin{aligned} p_n(t;z) &:= (S(t)\chi(z))_n \\ &= \frac{1}{\sqrt{\tau_n(t)\tau_{n+1}(t)}} \det \left(\begin{array}{ccc|c} & m_n(t) & & \begin{matrix} 1 \\ z \\ \vdots \end{matrix} \\ \hline \mu_{n,0} & \dots & \mu_{n,n-1} & z^n \end{array}\right) \\ &= z^n h_n^{-1/2} \frac{\tau_n(t - [z^{-1}])}{\tau_n(t)}, \qquad h_n := \frac{\tau_{n+1}(t)}{\tau_n(t)} \end{aligned} \tag{3.1.9}$$

- $q(t,z) = (q_n(t;z))_{n\geq 0}$, *with* $q_n(t;z) := z\int_{\mathbb{R}^n} \frac{p_n(t;u)}{z-u}\rho_t(u)du$, *satisfying* $(L(t)q(t;z))_n = zq_n(t;z)$, $n \geq 1$; $q_n(t;z)$ *enjoys the following representations:*

$$\begin{aligned} q_n(t;z) := z\int_{\mathbb{R}^n} \frac{p_n(t;u)}{z-u}\rho_t(u)du &= \left(S^{\top -1}(t)\chi(z^{-1})\right)_n \\ &= \left(S(t)m_\infty(t)\chi(z^{-1})\right)_n \\ &= z^{-n}h_n^{-1/2}\frac{\tau_{n+1}(t+[z^{-1}])}{\tau_n(t)}. \end{aligned} \tag{3.1.10}$$

In the case $\beta = 2$, the Virasoro generators (2.1.11) take on a particularly elegant form, namely for $n \geq 0$,

$$\begin{aligned} \mathbb{J}^{(2)}_{k,n}(t) &= \sum_{i+j=k} :\mathbb{J}^{(1)}_{i,n}(t)\,\mathbb{J}^{(1)}_{j,n}(t): \;=\; J^{(2)}_k(t) + 2nJ^{(1)}_k(t) + n^2\delta_{0k} \\ \mathbb{J}^{(1)}_{k,n}(t) &= J^{(1)}_k(t) + n\delta_{0k}, \end{aligned}$$

[15] $()_{\mathfrak{s}}$ means: take the skew-symmetric part of (); for more details, see subsection 3.1.2.
[16] The explicit dependence on the boundary points c will be omitted in this point (iv).

with[17]

$$J_k^{(1)}=\frac{\partial}{\partial t_k}+\frac{1}{2}(-k)t_{-k}\ ,\ J_k^{(2)}=\sum_{i+j=k}\frac{\partial^2}{\partial t_i\partial t_j}+\sum_{-i+j=k}it_i\frac{\partial}{\partial t_j}+\frac{1}{4}\sum_{-i-j=k}it_ijt_j. \quad (3.1.11)$$

Statement (i) is already contained in section 2, whereas statement (ii) will be established in subsection 3.1.2, using elementary methods.

3.1.2 Sketch of Proof

Orthogonal polynomials and τ-function representation: The representation (3.1.4) of the determinants of moment matrices as integrals follows immediately from the fact that the square of a Vandermonde determinant can be represented as a sum of determinants

$$\Delta^2(u_1,\dots,u_n)=\sum_{\sigma\in S_n}\det\left(u_{\sigma(k)}^{\ell+k-2}\right)_{1\le k,\ell\le n}.$$

Indeed,

$$\begin{aligned}
n!\tau_n(t) &= n!\det m_n(t)\\
&= \sum_{\sigma\in S_n}\det\left(\int_E z_{\sigma(k)}^{\ell+k-2}\rho_t(z_{\sigma(k)})dz_{\sigma(k)}\right)_{1\le k,\ell\le n}\\
&= \sum_{\sigma\in S_n}\int_{E^n}\det\left(z_{\sigma(k)}^{\ell+k-2}\right)_{1\le k,\ell\le n}\rho_t(z_{\sigma(k)})dz_{\sigma(k)}\\
&= \int_{E^n}\Delta_n^2(z)\prod_{k=1}^n\rho_t(z_k)dz_k,
\end{aligned}$$

whereas the representation (3.1.4) in terms of integrals over Hermitian matrices follows from section 1.1.

The Borel factorization of m_∞ is responsible for the orthonormality of the polynomials $p_n(t;z)=(S(t)\chi(z))_n$; indeed,

$$\langle p_k(t;z),p_\ell(t;z)\rangle_{0\le k,\ell<\infty}=\int_E S\chi(z)(S\chi(z))^\top\rho_t(z)dz=Sm_\infty S^\top=I.$$

Note that $S\chi(z)(S\chi(z))^\top$ should be viewed as a semi-infinite matrix obtained by multiplying a semi-infinite column and row. The determinantal representation (3.1.9) follows at once from noticing that $\langle p_n(t;z),z^k\rangle=0$ for $0\le k\le n-1$, because taking that inner-product produces two identical columns in the matrix thus obtained. From the same representation (3.1.9), one has $p_n(t;z)=h_n^{-1}z^n+\dots$, where $h_n:=(\tau_{n+1}/\tau_n(t))^{1/2}$.

[17] The expression $J_k^{(1)}=0$ for $k=0$.

The "Sato" representation (3.1.9) of $p_n(t;z)$ in terms of the determinant $\tau_n(t)$ of the moment matrix can be shown by first proving the Heine representation of the orthogonal polynomials, which goes as follows:

$h_n p_n(t;z)$

$$= \frac{1}{\tau_n}\det\left(\begin{array}{ccc|c} & & & 1 \\ & m_n(t) & & z \\ & & & \vdots \\ \hline \mu_{n,0} & \cdots & \mu_{n,n-1} & z^n \end{array}\right)$$

$$= \frac{1}{\tau_n}\int_{E^n}\det\left(\begin{array}{cccc|c} u_1^0 & u_2^1 & \cdots & u_n^{n-1} & 1 \\ u_1^1 & u_2^2 & \cdots & u_n^n & z \\ \vdots & & & \vdots & \vdots \\ u_1^{n-1} & u_2^n & \cdots & u_n^{2n-2} & z^{n-1} \\ \hline u_1^n & u_2^{n+1} & \cdots & u_n^{2n-1} & z^n \end{array}\right)\prod_1^n \rho_t(u_i)du_i$$

$$= \frac{1}{\tau_n}\int_{E^n}\det\left(\begin{array}{cccc|c} u_1^0 & u_2^0 & \cdots & u_n^0 & 1 \\ u_1^1 & u_2^1 & \cdots & u_n^1 & z \\ \vdots & & & \vdots & \vdots \\ u_1^{n-1} & u_2^{n-1} & \cdots & u_n^{n-1} & z^{n-1} \\ \hline u_1^n & u_2^n & \cdots & u_n^n & z^n \end{array}\right) u_1^0 u_2^1\cdots u_n^{n-1}\prod_1^n \rho_t(u_i)du_i$$

$$= \frac{1}{\tau_n}\int_{E^n}\det\left(\begin{array}{cccc|c} u_{\sigma(1)}^0 & u_{\sigma(2)}^0 & \cdots & u_{\sigma(n)}^0 & 1 \\ u_{\sigma(1)}^1 & u_{\sigma(2)}^1 & \cdots & u_{\sigma(n)}^1 & z \\ \vdots & & & \vdots & \vdots \\ u_{\sigma(1)}^{n-1} & u_{\sigma(2)}^{n-1} & \cdots & u_{\sigma(n)}^{n-1} & z^{n-1} \\ \hline u_{\sigma(1)}^n & u_{\sigma(2)}^n & \cdots & u_{\sigma(n)}^n & z^n \end{array}\right)$$

$$u_{\sigma(1)}^0 u_{\sigma(2)}^1\cdots u_{\sigma(n)}^{n-1}\prod_1^n \rho_t(u_{\sigma(i)})du_{\sigma(i)}\ ,$$

for any permutation $\sigma \in S_n$

$$= \frac{1}{\tau_n}\int_{E^n}\det\left(\begin{array}{cccc|c} u_1^0 & u_2^0 & \cdots & u_n^0 & 1 \\ u_1^1 & u_2^1 & \cdots & u_n^1 & z \\ \vdots & & & \vdots & \vdots \\ u_1^{n-1} & u_2^{n-1} & \cdots & u_n^{n-1} & z^{n-1} \\ \hline u_1^n & u_2^n & \cdots & u_n^n & z^n \end{array}\right)$$

$$(-1)^\sigma u_{\sigma(1)}^0 u_{\sigma(2)}^1\cdots u_{\sigma(n)}^{n-1}\prod_1^n \rho_t(u_i)du_i$$

$$= \frac{1}{n!\tau_n}\int_{E^n}\Delta_n^2(u)\prod_{k=1}^n (z-u_k)\rho_t(u_k)du_k, \quad \text{upon summing over all } \sigma.$$

Therefore, using again the representation of $\Delta^2(z)$ as a sum of determinants, Heine's formula leads to

$$\begin{aligned}
&h_n p_n(t,z)\\
&= \frac{z^n}{n!\tau_n}\int_{E^n}\sum_{\sigma\in S_n}\det\left(u_{\sigma(k)}^{\ell+k-2}\right)_{1\le k,\ell\le n}\prod_{k=1}^{n}\left(1-\frac{u_{\sigma(k)}}{z}\right)\rho_t(u_{\sigma(k)})du_{\sigma(k)},\\
&= \frac{z^n}{n!\tau_n}\int_{E^n}\sum_{\sigma\in S_n}\det\left(u_{\sigma(k)}^{\ell+k-2}-\frac{1}{z}u_{\sigma(k)}^{\ell+k-1}\right)_{1\le k,\ell\le n}\rho_t(u_{\sigma(k)})du_{\sigma(k)}\\
&= \frac{z^n}{\tau_n}\det\left(\mu_{ij}-\frac{1}{z}\mu_{i,j+1}\right)_{0\le i,j\le n-1}\\
&= \frac{z^n}{\tau_n}\det\left(\mu_{ij}(t-[z^{-1}])\right)_{0\le i,j\le n-1}\\
&= z^n\frac{\tau_n(t-[z^{-1}])}{\tau_n(t)}, \qquad (3.1.12)
\end{aligned}$$

invoking the fact that

$$\begin{aligned}
\mu_{ij}(t-[z^{-1}]) = \int u^{i+j}e^{\sum_1^\infty\left(t_i-\frac{z^{-i}}{i}\right)u^i}\rho(u)du &= \int u^{i+j}\left(1-\frac{u}{z}\right)\rho(u)du\\
&= \mu_{i+j}-\frac{1}{z}\mu_{i+j+1}.
\end{aligned}$$

Formula (3.1.10) follows from computing on the one hand $S(t)m_\infty\chi(z)$ using the explicit moments μ_{ij}, together with (3.1.12), and on the other hand the equivalent expression $S^{\top-1}(t)\chi(z^{-1})$. Indeed, using $(S(t)\chi(z))_n = p_n(t;z) = \sum_0^n p_{nk}(t)z^k$,

$$\begin{aligned}
\sum_{j\ge0}(Sm_\infty)_{nj}z^{-j} &= \sum_{j\ge0}z^{-j}\sum_{\ell\ge0}p_{n\ell}(t)\mu_{\ell j}\\
&= \sum_{j\ge0}z^{-j}\sum_{\ell\ge0}p_{n\ell}(t)\int_E u^{\ell+j}\rho_t(u)du\\
&= \int_E\sum_{\ell\ge0}p_{n\ell}(t)u^\ell\sum_{j\ge0}\left(\frac{u}{z}\right)^j\rho_t(u)du\\
&= z\int_E\frac{p_n(t,u)\rho_t(u)}{z-u}du.
\end{aligned}$$

Mimicking computation (3.1.12), one shows

$$h_n\sum_{j\ge0}\left(S^{\top-1}(t)\right)_{nj}z^{-j} = \frac{\tau_{n+1}(t+[z^{-1}])}{\tau_n(t)}z^{-n},$$

from which (3.1.10) follows, upon using $Sm_\infty = S^{\top-1}$. Details of this and subsequent derivation can be found in [5, 6].

The vectors p and q are eigenvectors of L. Indeed, remembering $\chi(z) = (1, z, z^2, ...)^\top$, and the shift $(\Lambda v)_n = v_{n+1}$, we have

$$\Lambda\chi(z) = z\chi(z) \text{ and } \Lambda^\top\chi(z^{-1}) = z\chi(z^{-1}) - ze_1, \quad \text{with } e_1 = (1, 0, 0, ...)^\top.$$

Therefore, $p(z) = S\chi(z)$ and $q(z) = S^{\top-1}\chi(z^{-1})$ are eigenvectors, in the sense

$$\begin{aligned} Lp &= S\Lambda S^{-1}S\chi(z) = zS\chi(z) = zp \\ L^\top q &= S^{\top-1}\Lambda^\top S^\top S^{\top-1}\chi(z^{-1}) = zS^{\top-1}\chi(z^{-1}) - zS^{\top-1}e_1 = zq - zS^{\top-1}e_1. \end{aligned}$$

Then, using $L = L^\top$, one is lead to

$$((L - zI)p)_n = 0, \quad \text{for } n \geq 0 \quad \text{and} \quad ((L - zI)q)_n = 0, \quad \text{for } \ n \geq 1.$$

Toda lattice and Lie algebra splitting: The Lie algebra splitting of semi-infinite matrices and the corresponding projections (used in (3.1.8)), denoted by $(\ )_\mathfrak{s}$ and $(\ )_\mathfrak{b}$ are defined as follows:

$$gl(\infty) = \mathfrak{s} \oplus \mathfrak{b} \left\{ \begin{array}{l} \mathfrak{s} = \{\text{skew-symmetric matrices}\} \\ \mathfrak{b} = \{\text{lower-triangular matrices}\} \end{array} \right. .$$

Conjugating the shift matrix Λ by $S(t)$ yields a matrix

$$\begin{aligned} L(t) &= S(t)\Lambda S(t)^{-1} \\ &= S\Lambda S^{-1}S^{\top-1}S^\top \\ &= S\Lambda m_\infty S^\top, \text{ using (3.1.3)}, \\ &= Sm_\infty\Lambda^\top S^\top, \text{ using } \Lambda m_\infty = m_\infty\Lambda^\top, \\ &= S(S^{-1}S^{\top-1})\Lambda^\top S^\top, \text{ using again (3.1.3)}, \\ &= (S\Lambda S^{-1})^\top = L(t)^\top, \end{aligned}$$

which is symmetric and thus tridiagonal. Moreover, from (3.1.3) one computes

$$\begin{aligned} 0 &= S\left(\Lambda^k m_\infty - \frac{\partial m_\infty}{\partial t_k}\right)S^\top \\ &= S\Lambda^k S^{-1} - S\frac{\partial}{\partial t_k}(S^{-1}S^{\top-1})S^\top \\ &= L^k + \frac{\partial S}{\partial t_k}S^{-1} + S^{\top-1}\frac{\partial S^\top}{\partial t_k}. \end{aligned}$$

Upon taking the $()_-$ and $()_0$ parts of this equation (A_- means the lower-triangular part of the matrix A, including the diagonal and A_0 the diagonal part) leads to

$$(L^k)_- + \frac{\partial S}{\partial t_k}S^{-1} + \left(S^{\top-1}\frac{\partial S^\top}{\partial t_k}\right)_0 = 0 \text{ and } \left(\frac{\partial S}{\partial t_k}S^{-1}\right)_0 = -\frac{1}{2}(L^k)_0.$$

Upon observing that for any symmetric matrix

$$\begin{pmatrix} a & c \\ c & b \end{pmatrix}_{\mathfrak{b}} = \begin{pmatrix} a & 0 \\ 2c & b \end{pmatrix} = 2\begin{pmatrix} a & c \\ c & b \end{pmatrix}_{-} - \begin{pmatrix} a & c \\ c & b \end{pmatrix}_{0},$$

it follows that the matrices $L(t)$, $S(t)$ and the vector $p(t;z) = (p_n(t;z))_{n\geq 0} = S(t)\chi(z)$ satisfy the (commuting) differential equations and the eigenvalue problem

$$\frac{\partial S}{\partial t_k} = -\frac{1}{2}(L^k)_{\mathfrak{b}} S, \qquad L(t)p(t;z) = zp(t;z), \tag{3.1.13}$$

and thus

$$\frac{\partial L}{\partial t_k} = -\left[\frac{1}{2}(L^k)_{\mathfrak{b}}, L\right], \quad \frac{\partial p}{\partial t_k} = -\frac{1}{2}(L^k)_{\mathfrak{b}} p.$$

(Standard Toda lattice)

The bilinear identity: The functions $\tau_n(t)$ satisfy the following identity, for $n \geq m+1$, $t, t' \in \mathbb{C}$, where one integrates along a small circle about ∞,

$$\oint_{z=\infty} \tau_n(t - [z^{-1}])\tau_{m+1}(t' + [z^{-1}])e^{\sum(t_i - t_i')z^i} z^{n-m-1} dz = 0. \tag{3.1.14}$$

An elementary proof can be given by expressing the left hand side of (3.1.14), in terms of $p_n(t;z)$ and $p_m(t,z)$, using (3.1.9) and (3.1.10). One uses below the following identity (see [2])

$$\int_{\mathbb{R}} f(z)g(z)dz = \left\langle f, \int_{\mathbb{R}} \frac{g(u)}{z-u} du \right\rangle_{\infty}, \tag{3.1.15}$$

involving the residue pairing[18]. So, modulo terms depending on t and t' only, the left hand side of (3.1.14) equals

$$\begin{aligned} &\oint_{z=\infty} dz\ z^{-n} p_n(t;z) e^{\sum_1^\infty (t_i - t_i')z^i} z^{n-m-1} z^{m+1} \int_{\mathbb{R}} \frac{p_m(t';u)}{z-u} e^{\sum_1^\infty t_i' u^i} \rho(u) du \\ &= \int_{\mathbb{R}} p_n(t;z) e^{\sum(t_i - t_i')z^i} p_m(t';z) e^{\sum t_i' z^i} \rho(z) dz, \text{ using (3.1.15)}, \\ &= \int_{\mathbb{R}} p_n(t;z) p_m(t';z) e^{\sum t_i z^i} \rho(z) dz = 0, \text{ when } m \leq n-1. \end{aligned} \tag{3.1.16}$$

[18] The residue pairing about $z = \infty$ between $f = \sum_{i\geq 0} a_i z^i \in \mathcal{H}^+$ and $g = \sum_{j\in\mathbb{Z}} b_j z^{-j-1} \in \mathcal{H}$ is defined as:

$$\langle f, g\rangle_\infty = \oint_{z=\infty} f(z)g(z)\frac{dz}{2\pi i} = \sum_{i\geq 0} a_i b_i.$$

The KP-hierarchy: Setting $n=m+1$, shifting $t\mapsto t-y, t'\mapsto t+y$, evaluating the residue and Taylor expanding in y_k and using the Schur polynomials s_n, leads to (see footnote 14 for the definition of $p(\partial_t)f\circ g$.)

$$\begin{aligned}
0 &= \frac{1}{2\pi i}\oint dz\; e^{-\sum_1^\infty 2y_i z^i}\tau_n(t-y-[z^{-1}])\tau_n(t+y+[z^{-1}]) \\
&= \frac{1}{2\pi i}\oint dz\;\left(\sum_0^\infty z^i s_i(-2y)\right)\left(\sum_0^\infty z^{-j}s_j(\tilde\partial)\right)e^{\sum_1^\infty y_k\frac{\partial}{\partial t_k}}\tau_n\circ\tau_n \\
&= e^{\sum_1^\infty y_k\frac{\partial}{\partial t_k}}\sum_0^\infty s_i(-2y)s_{i+1}(\tilde\partial)\tau_n\circ\tau_n \\
&= \left(1+\sum_1^\infty y_j\frac{\partial}{\partial t_j}+O(y^2)\right)\left(\frac{\partial}{\partial t_1}+\sum_1^\infty s_{i+1}(\tilde\partial)(-2y_i+O(y^2))\right)\tau_n\circ\tau_n \\
&= \left(\frac{\partial}{\partial t_1}+\sum_1^\infty y_k\left(\frac{\partial}{\partial t_k}\frac{\partial}{\partial t_1}-2s_{k+1}(\tilde\partial)\right)\right)\tau_n\circ\tau_n+O(y^2),
\end{aligned}$$

thus yielding (3.1.6), taking into acount the fact that $(\partial/\partial t_1)\tau\circ\tau=0$ and the coefficient of y_k is trivial for $k=1,2$.

The Riemann-Hilbert problem: Observe that, as a function of z, the integral (3.1.10) has a jump accross the real axis

$$\frac{1}{2\pi i}\lim_{\substack{z'\to z\\ \mathcal{I}z'<0}}\int_{\mathbb{R}}\frac{p_n(t;u)}{z'-u}\rho_t(u)du = p_n(t,z)\rho_t(z)+\frac{1}{2\pi i}\lim_{\substack{z'\to z\\ \mathcal{I}z'>0}}\int_{\mathbb{R}}\frac{p_n(t;u)}{z'-u}\rho_t(u)du,$$

and thus we have: (see [29, 19, 5])

Corollary 3.2 *The matrix*

$$Y_n(z)=\begin{pmatrix}\frac{\tau_n(t-[z^{-1}])}{\tau_n(t)}z^n & \frac{\tau_{n+1}(t+[z^{-1}])}{\tau_n(t)}z^{-n-1}\\[2ex] \frac{\tau_{n-1}(t-[z^{-1}])}{\tau_n(t)}z^{n-1} & \frac{\tau_n(t+[z^{-1}])}{\tau_n(t)}z^{-n}\end{pmatrix}$$

satisfies the Riemann–Hilbert problem[19]*:*

1. *$Y_n(z)$ holomorphic on the $\mathbb{C}_+$ and $\mathbb{C}_-$*
2. *$Y_{n-}(z)=Y_{n+}(z)\begin{pmatrix}1 & 2\pi i\rho_t(z)\\ 0 & 1\end{pmatrix}$ (Jump condition)*
3. *$Y_n'(z)\begin{pmatrix}z^{-n} & 0\\ 0 & z^n\end{pmatrix}=1+O(z^{-1})$, when $z\to\infty$.*

[19] $\mathbb{C}_+$ and $\mathbb{C}_-$ denote the Siegel upper and lower half plane.

3.2 Pfaff Lattice and symmetric/symplectic matrix integrals

3.2.1 Pfaff lattice, factorization of skew-symmetric matrices and skew-orthogonal polynomials

Consider an inner-product, with a skew-symmetric weight $\tilde{\rho}(y,z)$,

$$\langle f,g\rangle_t = \int\!\!\int_{\mathbb{R}^2} f(y)g(z)e^{\sum t_i(y^i+z^i)}\tilde{\rho}(y,z)dy\,dz, \text{ with } \tilde{\rho}(z,y) = -\tilde{\rho}(y,z). \tag{3.2.1}$$

Since $\langle f,g\rangle_t = -\langle g,f\rangle_t$, the moment matrix, depending on $t=(t_1,t_2,\dots)$,

$$m_n(t) = (\mu_{ij}(t))_{0\le i,j\le n-1} = (\langle y^i, z^j\rangle_t)_{0\le i,j\le n-1}$$

is skew-symmetric. It is clear from formula (3.2.1) that the semi-infinite matrix m_∞ evolves in t according to the *commuting vector fields*:

$$\frac{\partial \mu_{ij}}{\partial t_k} = \mu_{i+k,j} + \mu_{i,j+k}, \text{ i.e., } \frac{\partial m_\infty}{\partial t_k} = \Lambda^k m_\infty + m_\infty \Lambda^{\top k}. \tag{3.2.2}$$

Since m_∞ is skew-symmetric, m_∞ does not admit a Borel factorization in the standard sense, but m_∞ admits a unique factorization, with an inserted semi-infinite, skew-symmetric matrix J, with $J^2=-I$, of the form (1.1.12): (see [2])

$$m_\infty(t) = Q^{-1}(t) J\, Q^{\top -1}(t),$$

where

$$Q(t) = \left(\begin{array}{cccc}
\ddots & & & 0 \\
 & & 0 & \\
\boxed{\begin{array}{cc} Q_{2n,2n} & 0 \\ 0 & Q_{2n,2n}\end{array}} & & & \\
 & \boxed{\begin{array}{cc} Q_{2n+2,2n+2} & 0 \\ 0 & Q_{2n+2,2n+2}\end{array}} & & \\
* & & & \\
 & & & \ddots
\end{array}\right) \in K. \tag{3.2.3}$$

K is the group of lower-triangular invertible matrices of the form above, with Lie algebra $\mathfrak{k}$. Consider the Lie algebra splitting, given by

$$gl(\infty) = \mathfrak{k} \oplus \mathfrak{n} \left\{ \begin{array}{l} \mathfrak{k} = \{\text{lower-triangular matrices of the form (3.2.3)}\} \\ \mathfrak{n} = sp(\infty) = \{a \text{ such that } Ja^\top J = a\}, \end{array}\right.$$

(3.2.4)

with unique decomposition[20]

$$
\begin{aligned}
a &= (a)_{\mathfrak{k}} + (a)_{\mathfrak{n}} \\
&= \left((a_- - J(a_+)^\top J) + \frac{1}{2}(a_0 - J(a_0)^\top J) \right) \\
&\qquad + \left((a_+ + J(a_+)^\top J) + \frac{1}{2}(a_0 + J(a_0)^\top J) \right). \qquad (3.2.5)
\end{aligned}
$$

Consider as a special skew-symmetric weight (3.2.1): (see [13])

$$
\tilde\rho(y,z) = 2D^\alpha \delta(y-z)\tilde\rho(y)\tilde\rho(z) \text{ , with } \alpha = \mp 1, \quad \tilde\rho(y) = e^{-\tilde V(y)}, \qquad (3.2.6)
$$

together with the associated inner-product[21] of type (3.2.1):

$$
\begin{aligned}
\langle f, g\rangle_t &= \int\!\!\int_{\mathbb{R}^2} f(y)g(z)e^{\sum t_i(y^i+z^i)} 2D^\alpha \delta(y-z)\tilde\rho(y)\tilde\rho(z)dy\,dz \qquad (3.2.7) \\
&= \begin{cases} \displaystyle\int\!\!\int_{\mathbb{R}^2} f(y)g(z)e^{\sum_1^\infty t_i(y^i+z^i)} \varepsilon(y-z)\tilde\rho(y)\tilde\rho(z)dy\,dz, & \text{for } \alpha = -1 \\ \displaystyle\int_{\mathbb{R}} \{f,g\}(y)e^{\sum_1^\infty 2t_i y^i}\tilde\rho(y)^2 dy, & \text{for } \alpha = +1 \end{cases}
\end{aligned}
$$

in terms of the Wronskian $\{f,g\} := \frac{\partial f}{\partial y} g - f\frac{\partial g}{\partial y}$. The moments with regard to these inner-products (with that precise definition of time t!) satisfy the differential equations $\partial\mu_{ij}/\partial t_k = \mu_{i+k,j} + \mu_{i,j+k}$, as in (3.2.2).

It is well known that the determinant of an odd skew-symmetric matrix equals 0, whereas the determinant of an even skew-symmetric matrix is the square of a polynomial in the entries, the *Pfaffian* [22]. Define now the *"Pfaffian τ-functions"*, defined with regard

[20] $a_\pm$ refers to projection onto strictly upper (strictly lower) triangular matrices, with all 2×2 diagonal blocks equal zero. a_0 refers to projection onto the "diagonal", consisting of 2×2 blocks.

[21] $\varepsilon(x) = \text{sign } x$, having the property $\varepsilon' = 2\delta(x)$.

[22] with a sign specified below. So $\det(m_{2n-1}(t)) = 0$ and

$$
\begin{aligned}
(\det m_{2n}(t))^{1/2} &= pf(m_{2n}(t)) \\
&= \frac{1}{n!}(dx_0 \wedge dx_1 \wedge \ldots \wedge dx_{2n-1})^{-1} \left(\sum_{0\le i<j\le 2n-1} \mu_{ij}(t) dx_i \wedge dx_j \right)^n .
\end{aligned}
$$

to the inner-products (3.2.7) :

$$\tau_{2n}(t) := \begin{cases} pf\left(\iint_{\mathbb{R}^2} y^k z^\ell \varepsilon(y-z) e^{\sum_1^\infty t_i(y^i+z^i)} \tilde\rho(y)\tilde\rho(z)dydz\right)_{0\leq k,\ell\leq 2n-1}, \alpha=-1 \\ pf\left(\int_{\mathbb{R}} \{y^k,y^\ell\} e^{\sum_1^\infty 2t_iy^i}\tilde\rho^2(y)dy\right)_{0\leq k,\ell\leq 2n-1}, \qquad \alpha=+1 \end{cases} \tag{3.2.8}$$

Setting

$$\begin{cases} \tilde\rho(z)=\rho(z)I_E(z) & \text{for } \alpha=-1 \\ \tilde\rho(z)=\rho^{1/2}(z)I_E(z),\ t\to t/2 & \text{for } \alpha=+1, \end{cases}$$

in the identities (3.2.8) leads to the identities (3.2.9) between integrals and Pfaffians, spelled out in Theorem 3.3 below. Remember $c=(c_1,...,c_{2r})$ stands for the boundary points of the disjoint union $E\subset\mathbb{R}$. $\mathcal{S}_n(E)$ and $\mathcal{T}_n(E)$ denotes the set of symmetric/symplectic ensemble with spectrum in E.

Theorem 3.3 (Adler-Horozov-van Moerbeke [9], Adler-van Moerbeke [7]) *The following integrals $I_n(t,c)$ are Pfaffians:*

$$\begin{aligned} I_n &= \int_{E^n} |\Delta_n(z)|^\beta \prod_{k=1}^n \left(e^{\sum_1^\infty t_iz_k^i}\rho(z_k)dz_k\right) \\ &= \begin{cases} \underline{\beta=1} \\ \quad = \int_{\mathcal{S}_n(E)} e^{\mathrm{Tr}(-V(X)+\sum_1^\infty t_iX^i)}dX, \ \underline{(n\ even)} \\ \quad = n!pf\left(\iint_{E^2} y^kz^\ell\varepsilon(y-z)e^{\sum_1^\infty t_i(y^i+z^i)}\rho(y)\rho(z)dydz\right)_{0\leq k,\ell\leq n-1} \\ \quad = n!\tau_n(t,c) \\ \\ \underline{\beta=4} \\ \quad = \int_{\mathcal{T}_{2n}(E)} e^{\mathrm{Tr}(-V(X)+\sum_1^\infty t_iX^i)}dX, \ \underline{(n\ arbitrary)} \\ \quad = n!pf\left(\int_E \{y^k,y^\ell\}e^{\sum_1^\infty t_iy^i}\rho(y)dy\right)_{0\leq k,\ell\leq 2n-1} \\ \quad = n!\tau_{2n}(t/2,c). \end{cases} \end{aligned} \tag{3.2.9}$$

The I_n and τ_n's satisfy

(i) *The* *<u>Virasoro constraints</u>*[23] *(2.1.7) for $\beta=1,4$,*

$$\left(-\sum_1^{2r} c_i^{k+1}f(c_i)\frac{\partial}{\partial c_i} + \sum_{i\geq 0}\left(a_i\ {}^\beta\mathbb{J}^{(2)}_{k+i,n} - b_i\ {}^\beta\mathbb{J}^{(1)}_{k+i+1,n}\right)\right) I_n(t,c)=0. \tag{3.2.10}$$

[23] here the a_i's and b_i's are defined in the usual way, in terms of $\rho(z)$; namely, $-\frac{\rho'}{\rho}=\frac{\sum b_iz^i}{\sum a_iz^i}$

(ii) *The Pfaff-KP hierarchy: (see footnote 14 for notation)*

$$\left(\mathbf{s}_{k+4}(\tilde{\partial})-\frac{1}{2}\frac{\partial^2}{\partial t_1\partial t_{k+3}}\right)\tau_n\circ\tau_n=\mathbf{s}_k(\tilde{\partial})\ \tau_{n+2}\circ\tau_{n-2} \tag{3.2.11}$$

n *even*, $k=0,1,2,\ldots$.

of which the first equation reads (n even)

$$\left(\left(\frac{\partial}{\partial t_1}\right)^4+3\left(\frac{\partial}{\partial t_2}\right)^2-4\frac{\partial^2}{\partial t_1\partial t_3}\right)\log\tau_n+6\left(\frac{\partial^2}{\partial t_1^2}\log\tau_n\right)^2=12\frac{\tau_{n-2}\tau_{n+2}}{\tau_n^2}.$$

(iii) *The Pfaff Lattice: The time-dependent matrix $L(t)$, zero above the first superdiagonal, obtained by dressing up Λ,*

$$L(t)=Q(t)\Lambda Q(t)^{-1}=\begin{pmatrix} * & 1 & & & & \\ & * & (h_2/h_0)^{1/2} & & O & \\ & & * & 1 & & \\ & & & * & (h_4/h_2)^{1/2} & \\ & & & & * & \\ & * & & & & \ddots \\ & & & & & \end{pmatrix} \tag{3.2.12}$$

satisfies the Hamiltonian commuting equations

$$\frac{\partial L}{\partial t_i}=[-(L^i)_{\mathfrak{k}},L]. \qquad \textbf{(Pfaff lattice)} \tag{3.2.13}$$

(iv) *Skew-orthogonal polynomials: The vector of time-dependent polynomials $q(t;z):=(q_n(t;z))_{n\geq 0}=Q(t)\chi(z)$ in z satisfy the eigenvalue problem*

$$L(t)q(t,z)=zq(t,z) \tag{3.2.14}$$

and enjoy the following representations (with $h_{2n} = \frac{\tau_{2n+2}(t)}{\tau_{2n}(t)}$)

$$
\begin{aligned}
q_{2n}(t;z) &= \frac{h_{2n}^{-1/2}}{\tau_{2n}(t)} pf \left(\begin{array}{cccc|c} & m_{2n+1}(t) & & & \begin{matrix} 1 \\ z \\ \vdots \\ z^{2n} \end{matrix} \\ \hline -1 & -z & \dots & -z^{2n} & 0 \end{array} \right) \\
&= z^{2n} h_{2n}^{-1/2} \frac{\tau_{2n}(t-[z^{-1}])}{\tau_{2n}(t)} = z^{2n} h_{2n}^{-1/2} + \dots \quad , \\
q_{2n+1}(t;z) &= \frac{h_{2n}^{-1/2}}{\tau_{2n}(t)} pf \left(\begin{array}{ccc|cc} & m_{2n}(t) & & \begin{matrix} 1 \\ z \\ \vdots \\ z^{2n-1} \end{matrix} & \begin{matrix} \mu_{0,2n+1} \\ \mu_{1,2n+1} \\ \vdots \\ \mu_{2n-1,2n+1} \end{matrix} \\ \hline -1 & \dots & -z^{2n-1} & 0 & -z^{2n+1} \\ \mu_{2n+1,0} & \dots & \mu_{2n+1,2n-1} & z^{2n+1} & 0 \end{array} \right). \\
&= z^{2n} h_{2n}^{-1/2} \frac{1}{\tau_{2n}(t)} \left(z + \frac{\partial}{\partial t_1} \right) \tau_{2n}(t-[z^{-1}]) = z^{2n+1} h_{2n}^{-1/2} + \dots .
\end{aligned}
\tag{3.2.15}
$$

They are skew-orthogonal polynomials in z; i.e.,

$$\langle q_i(t;z), q_j(t;z) \rangle_t = J_{ij}.$$

The hierarchy (3.2.11) already appears in the work of Kac-van de Leur [42] in the context of, what they call the DKP-hierarchy, and interesting further work has been done by van de Leur [71].

3.2.2 Sketch of Proof

Skew-orthogonal polynomials and the Pfaff Lattice: The equalities (3.2.9) between the Pfaffians and the matrix integrals are based on two identities [49], the first one due to de Bruyn,

$$
\begin{aligned}
&\frac{1}{n!} \int_{\mathbb{R}^n} \prod_1^n dy_i \quad \det \Big(F_i(y_1) \quad G_i(y_1) \quad \dots \quad F_i(y_n) \quad G_i(y_n) \Big)_{0 \le i \le 2n-1} \\
&= \det{}^{1/2} \left(\int_{\mathbb{R}} (G_i(y) F_j(y) - F_i(y) G_j(y)) dy \right)_{0 \le i,j \le 2n-1}
\end{aligned}
$$

and (Mehta [50])

$$\Delta_n^4(x) = \det \left(x_1^i \quad (x_1^i)' \quad x_2^i \quad (x_2^i)' \ \dots \ x_n^i \quad (x_n^i)' \right)_{0 \le i \le 2n-1}.$$

On the one hand, (see Mehta [49]), setting in the calculation below $\rho_t(z)=\rho(z)e^{\sum t_i z^i}I_E(z)$ and

$$F_i(x):=\int_{-\infty}^{x} y^i\rho_t(y)dy \quad \text{and} \quad G_i(x):=F_i'(x)=x^i\rho_t(x),$$

one computes: $(\rho_t(z):=\rho(z)e^{\sum t_i z^i})$

$$\begin{aligned}
&\frac{1}{(2n)!}\int_{\mathbb{R}^{2n}}|\Delta_{2n}(z)|\prod_{i=1}^{2n}\rho_t(z_i)dz_i\\
&= \int_{-\infty<z_1<z_2<\cdots<z_{2n}<\infty}\det\left(z_{j+1}^i\rho_t(z_{j+1})\right)_{0\le i,j\le 2n-1}\prod_{i=1}^{2n}dz_i,\\
&= \int_{-\infty<z_2<z_4<\cdots<z_{2n}<\infty}\prod_{k=1}^{n}\rho_t(z_{2k})dz_{2k}\\
&\quad\det\left(\int_{-\infty}^{z_2}z_1^i\rho_t(z_1)dz_1\ ,\ z_2^i\ ,\dots,\ \int_{z_{2n-2}}^{z_{2n}}z_{2n-1}^i\rho_t(z_{2n-1})dz_{2n-1}\ ,\ z_{2n}^i\right)_{0\le i\le 2n-1}\\
&= \int_{-\infty<z_2<z_4<\cdots<z_{2n}<\infty}\prod_{k=1}^{n}\rho_t(z_{2k})dz_{2k}\\
&\quad\det\left(F_i(z_2)\ ,\ z_2^i\ ,\ F_i(z_4)-F_i(z_2)\ ,\ z_4^i\ ,\ \dots\ ,\ F_i(z_{2n})-F_i(z_{2n-2})\ ,\ z_{2n}^i\right)_{0\le i\le 2n-1}\\
&= \int_{-\infty<z_2<z_4<\cdots<z_{2n}<\infty}\prod_{1}^{n}dz_i\quad\det\Big(F_i(z_2)\ ,\ G_i(z_2)\ ,\ \dots,\ F_i(z_{2n})\ ,\ G_i(z_{2n})\Big)_{0\le i\le 2n-1},\\
&= \frac{1}{n!}\int_{\mathbb{R}^n}\prod_{1}^{n}dy_i\quad\det\Big(F_i(y_1)\ ,\ G_i(y_1)\ ,\ \dots,\ F_i(y_n)\ ,\ G_i(y_n)\Big)_{0\le i\le 2n-1},\\
&= \det{}^{1/2}\left(\int_{\mathbb{R}}(G_i(y)F_j(y)-F_i(y)G_j(y))dy\right)_{0\le i,j\le 2n-1}\\
&= pf\left(\iint_{E^2}y^k z^\ell\varepsilon(y-z)e^{\sum_1^\infty t_i(y^i+z^i)}\rho(y)\rho(z)dydz\right)_{0\le k,\ell\le 2n-1}=\tau_{2n}(t),
\end{aligned}$$

establishing the first equation of (3.2.9), taking into account the results in section 2.1.

On the other hand, upon setting,

$$F_j(x)=x^j\rho(x)e^{\sum t_i x^i} \quad \text{and} \quad G_j(x):=F_j'(x)=\left(x^j\rho(x)e^{\sum t_i x^i}\right)',$$

one computes

$$
\begin{aligned}
&\frac{1}{n!}\int_{\mathbb{R}^n}\prod_{1\le i,j\le n}(x_i-x_j)^4\prod_{k=1}^{n}\left(\rho^2(x_k)e^{2\sum_{i=1}^{\infty}t_ix_k^i}dx_k\right)\\
&\quad=\ \frac{1}{n!}\int_{\mathbb{R}^n}\prod_{k=1}^{n}\left(\rho^2(x_k)e^{2\sum t_ix_k^i}dx_k\right)\\
&\qquad\qquad\qquad\det\left(x_1^i\quad (x_1^i)'\quad x_2^i\quad (x_2^i)'\ \dots\ x_n^i\quad (x_n^i)'\right)_{0\le i\le 2n-1}\\
&\quad=\ \frac{1}{n!}\int_{\mathbb{R}^n}\prod_{1}^{n}dy_i\ \ \det\Big(F_i(y_1)\ \ G_i(y_1)\ \ \dots\ \ F_i(y_n)\ \ G_i(y_n)\Big)_{0\le i\le 2n-1},\\
&\quad=\ \det{}^{1/2}\left(\int_{\mathbb{R}}(G_i(y)F_j(y)-F_i(y)G_j(y))dy\right)_{0\le i,j\le 2n-1}\\
&\quad=\ pf\ \left(\int_E\{y^k,y^\ell\}e^{\sum_1^\infty 2t_iy^i}\rho^2(y)dy\right)_{0\le k,\ell\le 2n-1}=\tau_{2n}(t),
\end{aligned}
$$

establishing the second equation (3.2.9).

The skew-orthogonality of the polynomials $q_k(t;z)$ follows immediately from the skew-Borel decomposition of m_∞:

$$\langle q_k(t,y),q_\ell(t,z)\rangle_{k,\ell\ge 0}=Q(\langle y^i,z^j\rangle)_{i,j\ge 0}Q^\top=Q\,m_\infty Q^\top=J.$$

with the q_n's admitting the representation (3.2.15) in terms of the moments.

Using $L=Q\Lambda Q^{-1}$, $m_\infty=Q^{-1}JQ^{\top-1}$ and $J^2=-I$, one computes from the differential equations (3.2.2)

$$
\begin{aligned}
0\ &=\ Q\left(\Lambda^k m_\infty+m_\infty\Lambda^{\top k}-\frac{\partial m_\infty}{\partial t_k}\right)Q^\top\\
&=\ (Q\Lambda^kQ^{-1})J-(JQ^{\top-1}\Lambda^{\top k}Q^\top J)J+\frac{\partial Q}{\partial t_k}Q^{-1}J-(JQ^{-1\top}\frac{\partial Q^\top}{\partial t_k}J)J\\
&=\ \left(L^k+\frac{\partial Q}{\partial t_k}Q^{-1}\right)-J\left(L^k+\frac{\partial Q}{\partial t_k}Q^{-1}\right)^\top J.
\end{aligned}
$$

Then computing the +, - and the diagonal part (in the sense of (3.2.4) and (3.2.5)) of the expression leads to commuting Hamiltonian differential equations for Q, and thus for L and $q(t;z)$, confirming (3.2.13):

$$\frac{\partial Q}{\partial t_i}=-(L^i)_{\mathfrak{k}}Q,\quad \frac{\partial L}{\partial t_i}=[(L^i)_{\mathfrak{n}},L],\quad \frac{\partial q}{\partial t_i}=-(L^i)_{\mathfrak{k}}q.\ \text{(Pfaff lattice)}\tag{3.2.16}$$

The bilinear identities: For all $n,m\ge 0$, the τ_{2n}'s satisfy the following bilinear

identity

$$\oint_{z=\infty} \tau_{2n}(t-[z^{-1}])\tau_{2m+2}(t'+[z^{-1}])e^{\sum(t_i-t_i')z^i}z^{2n-2m-2}\frac{dz}{2\pi i}$$
$$+\oint_{z=0}\tau_{2n+2}(t+[z])\tau_{2m}(t'-[z])e^{\sum(t_i'-t_i)z^{-i}}z^{2n-2m}\frac{dz}{2\pi i}=0. \tag{3.2.17}$$

The differential equation (3.2.2) on the moment matrix m_∞ admits the following solution, which upon using the Borel decomposition $m_\infty = Q^{-1}J\,Q^{\top -1}$, leads to:

$$m_\infty(0)=e^{-\sum_1^\infty t_k\Lambda^k}m_\infty(t)e^{-\sum_1^\infty t_k\Lambda^{\top k}}=\left(Q(t)e^{\sum_1^\infty t_k\Lambda^k}\right)^{-1}J\left(Q(t)e^{\sum_1^\infty t_k\Lambda^k}\right)^{\top -1}, \tag{3.2.18}$$

and so the right hand side of (3.2.11) is independent of t; say, equal to the same expression with t replaced by t'. Upon rearrangement, one finds

$$\left(Q(t)e^{\sum t_k\Lambda^k}\right)\left(JQ(t')e^{\sum t_k'\Lambda^k}\right)^{-1}=\left(JQ(t)e^{\sum t_k\Lambda^k}\right)^{\top -1}\left(Q(t')e^{\sum t_k'\Lambda^k}\right)^{\top}$$

and therefore[24]

$$\oint_{z=\infty}\left(Q(t)\chi(z)\otimes(JQ(t'))^{\top -1}\chi(z^{-1})\right)e^{\sum_1^\infty(t_k-t_k')z^k}\frac{dz}{2\pi iz}$$
$$=\oint_{z=0}\left((JQ(t))^{\top -1}\chi(z)\otimes Q(t')\chi(z^{-1})\right)e^{\sum_1^\infty(t_k'-t_k)z^{-k}}\frac{dz}{2\pi iz}. \tag{3.2.19}$$

Setting $t-t'=[z_1^{-1}]+[z_2^{-1}]$ into the exponential leads to

$$\begin{aligned} e^{\sum_1^\infty(t_k-t_k')z^k} &= \left(1-\frac{z}{z_1}\right)^{-1}\left(1-\frac{z}{z_2}\right)^{-1}\\ e^{\sum_1^\infty(t_k'-t_k)z^{-k}} &= \left(1-\frac{1}{zz_1}\right)\left(1-\frac{1}{zz_2}\right)\end{aligned}$$

and somewhat enlarging the integration circle about $z=\infty$ to include the points z_1 and z_2, the integrand on the left hand side has poles at $z=z_1$ and z_2, whereas the integrand on the right hand side is holomorphic. Combining the identity obtained and the one, with $z_2 \nearrow \infty$, one finds a functional relation involving a function $\varphi(t;z)=1+O(z^{-1})$:

$$\frac{\varphi(t-[z_2^{-1}];z_1)}{\varphi(t;z_1)}=\frac{\varphi(t-[z_1^{-1}];z_2)}{\varphi(t;z_2)},\quad t\in\mathbb{C}^\infty, z\in\mathbb{C}.$$

[24]using $\Lambda\chi(z)=z\chi(z)$, $\Lambda^\top\chi(z)=z^{-1}\chi(z)$ and the following matrix identities (see [25])

$$U_1V_1=\oint_{z=\infty}U_1\chi(z)\otimes V_1^\top\chi(z^{-1})\frac{dz}{2\pi iz},\quad U_2V_2=\oint_{z=0}U_2\chi(z)\otimes V_2^\top\chi(z^{-1})\frac{dz}{2\pi iz}.$$

Such an identity leads, by a standard argument (see e.g. the appendix in [8]) to the existence of a function $\tau(t)$ such that

$$\varphi(t;z)=\frac{\tau(t-[z^{-1}])}{\tau(t)}.$$

This fact combined with the bilinear identity (3.2.19) leads to the bilinear identity (3.2.17).

The Pfaff-KP-hierarchy: Shifting $t\mapsto t-y, t'\mapsto t+y$ in (3.2.17), evaluating the residue and Taylor expanding in y_k leads to:

$$\begin{aligned}
&\frac{1}{2\pi i}\oint_{z=\infty} e^{-\sum_1^\infty 2y_i z^i}\tau_{2n}(t-y-[z^{-1}])\tau_{2m+2}(t+y+[z^{-1}])z^{2n-2m-2}dz\\
&\qquad+\frac{1}{2\pi i}\oint_{z=0} e^{\sum_1^\infty 2y_i z^{-i}}\tau_{2n+2}(t-y+[z])\tau_{2m}(t+y-[z])z^{2n-2m}dz\\
&\qquad=\frac{1}{2\pi i}\oint_{z=\infty}\sum_{j=0}^\infty z^j\mathbf{s}_j(-2y)e^{\sum -y_i\frac{\partial}{\partial t_i}}\sum_{k=0}^\infty z^{-k}\mathbf{s}_k(-\tilde\partial)\tau_{2n}\circ\tau_{2m+2}z^{2n-2m-2}dz\\
&\qquad\quad+\frac{1}{2\pi i}\oint_{z=0}\sum_{j=0}^\infty z^{-j}\mathbf{s}_j(2y)e^{\sum -y_i\frac{\partial}{\partial t_i}}\sum_{k=0}^\infty z^{k}\mathbf{s}_k(\tilde\partial)\tau_{2n+2}\circ\tau_{2m}z^{2n-2m}dz
\end{aligned}$$

$$\begin{aligned}
&=\sum_{j-k=-2n+2m+1}\mathbf{s}_j(-2y)e^{\sum -y_i\frac{\partial}{\partial t_i}}\mathbf{s}_k(-\tilde\partial)\tau_{2n}\circ\tau_{2m+2}\\
&\qquad+\sum_{k-j=-2n+2m-1}\mathbf{s}_j(2y)e^{\sum -y_i\frac{\partial}{\partial t_i}}\mathbf{s}_k(\tilde\partial)\tau_{2n+2}\circ\tau_{2m}\\
&=...+y_k\left(\left(\frac{1}{2}\frac{\partial}{\partial t_1}\frac{\partial}{\partial t_k}-\mathbf{s}_{k+1}(\tilde\partial)\right)\tau_{2n}\circ\tau_{2n}+\mathbf{s}_{k-3}(\tilde\partial)\tau_{2n+2}\circ\tau_{2n-2}\right)+...,
\end{aligned}$$

establishing the Pfaff-KP hierarchy (3.2.11), different from the usual KP hierarchy, because of the presence of a right hand side.

<u>*Remark:*</u> L admits the following representation in terms of τ, much in the style of (3.1.7),

$$L=h^{-1/2}\begin{pmatrix}\hat L_{00} & \hat L_{01} & 0 & 0 & \\ \hat L_{10} & \hat L_{11} & \hat L_{12} & 0 & \\ * & \hat L_{21} & \hat L_{22} & \hat L_{23} & \\ * & * & \hat L_{32} & \hat L_{33} & \dots\\ & & & \vdots & \end{pmatrix}h^{1/2},$$

with the 2×2 entries $\hat{L}_{ij}$ and h, being a zero matrix, except for 2×2 matrices along the diagonal:

$$h = \operatorname{diag}(h_0 I_2, h_2 I_2, h_4 I_2, \dots),\ \ h_{2n} = \tau_{2n+2}/\tau_{2n}.$$

For example ($^{\cdot} = \frac{\partial}{\partial t_1}$)

$$\hat{L}_{nn} := \begin{pmatrix} -(\log \tau_{2n})^{\cdot} & 1 \\ -\frac{s_2(\tilde{\partial})\tau_{2n}}{\tau_{2n}} - \frac{s_2(-\tilde{\partial})\tau_{2n+2}}{\tau_{2n+2}} & (\log \tau_{2n+2})^{\cdot} \end{pmatrix}$$

$$\hat{L}_{n,n+1} := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad \hat{L}_{n+1,n} := \begin{pmatrix} * & (\log \tau_{2n+2})^{\cdot\cdot} \\ * & * \end{pmatrix}.$$

3.3 2d-Toda lattice and coupled Hermitian matrix integrals

3.3.1 2d-Toda lattice, factorization of moment matrices and bi-orthogonal polynomials

Consider the inner-product,

$$\langle f, g\rangle_{t,s} = \iint_{E\subset\mathbb{R}^2} f(y)g(z)e^{\sum_1^\infty (t_i y^i - s_i z^i) + cyz} dy\, dz, \tag{3.3.1}$$

on a subset $E = E_1 \times E_2 := \cup_{i=1}^r [c_{2i-1}, c_{2i}] \times \cup_{i=1}^s [\tilde{c}_{2i-1}, \tilde{c}_{2i}] \subset F_1 \times F_2 \subset \mathbb{R}^2$. Define the customary moment matrix, depending on $t = (t_1, t_2, \dots),\ s = (s_1, s_2, \dots)$,

$$m_n(t,s) = (\mu_{ij}(t,s))_{0\le i,j\le n-1} = (\langle y^i, z^j\rangle_{t,s})_{0\le i,j\le n-1}$$

and its factorization in lower- times upper-triangular matrices

$$m_\infty(t,s) = S_1^{-1}(t,s)S_2(t,s). \tag{3.3.2}$$

Then m_∞ evolves in t, s according to the equations

$$\frac{\partial \mu_{ij}}{\partial t_k} = \mu_{i+k,j}, \quad \frac{\partial \mu_{ij}}{\partial s_k} = -\mu_{i,j+k},\ \text{i.e.,}\ \frac{\partial m_\infty}{\partial t_k} = \Lambda^k m_\infty, \quad \frac{\partial m_\infty}{\partial s_k} = -m_\infty \Lambda^{\top k}. \tag{3.3.3}$$

dM in the integral (3.3.4) denotes properly normalized Haar measure on $\mathcal{H}_n$.

Theorem 3.4 (Adler-van Moerbeke [4, 3]) *The integrals $I_n(t,s;c,\tilde{c})$, with $I_0 = 1$,*

$$\begin{aligned} \tau_n = \det m_n = {} & \frac{1}{n!} I_n = \frac{1}{n!} \iint_{E^n} \Delta_n(x)\Delta_n(y) \prod_{k=1}^n e^{\sum_1^\infty (t_i x_k^i - s_i y_k^i) + c x_k y_k} dx_k dy_k \\ = {} & \iint_{\mathcal{H}_n^2(E)} e^{c\,\mathrm{Tr}(M_1 M_2)} e^{\mathrm{Tr} \sum_1^\infty (t_i M_1^i - s_i M_2^i)} dM_1 dM_2, \end{aligned} \tag{3.3.4}$$

satisfy:

(i) *Virasoro constraints*[25] *(2.2.8) for* $k \geq -1$,

$$\left(-\sum_{i=1}^{r} c_i^{k+1}\frac{\partial}{\partial c_i}+J_{k,n}^{(2)}\right)\tau_n^E + c\,\mathbf{s}_{k+n}(\tilde\partial_t)\mathbf{s}_n(-\tilde\partial_s)\tau_1^E\circ\tau_{n-1}^E \;=\; 0$$
$$\left(-\sum_{i=1}^{s} \tilde c_i^{k+1}\frac{\partial}{\partial \tilde c_i}+\tilde J_{k,n}^{(2)}\right)\tau_n^E + c\,\mathbf{s}_{n}(\tilde\partial_t)\mathbf{s}_{k+n}(-\tilde\partial_s)\tau_1^E\circ\tau_{n-1}^E \;=\; 0, \tag{3.3.5}$$

with

$$J_{k,n}^{(2)} = \frac{1}{2}(J_k^{(2)}+(2n+k+1)J_k^{(1)}+n(n+1)J_k^{(0)}),$$
$$\tilde J_{k,n}^{(2)} = \frac{1}{2}(\tilde J_k^{(2)}+(2n+k+1)\tilde J_k^{(1)}+n(n+1)J_k^{(0)}).$$

(ii) *A Wronskian identity*[26]*:*

$$\left\{\frac{\partial^2\log\tau_n}{\partial t_1\partial s_2},\frac{\partial^2\log\tau_n}{\partial t_1\partial s_1}\right\}_{t_1}+\left\{\frac{\partial^2\log\tau_n}{\partial s_1\partial t_2},\frac{\partial^2\log\tau_n}{\partial t_1\partial s_1}\right\}_{s_1}=0. \tag{3.3.6}$$

(iii) *The 2d-Toda lattice: Given the factorization (3.3.2), the matrices* $L_1 := S_1\Lambda S_1^{-1}$ *and* $L_2 := S_2\Lambda^\top S_2^{-1}$*, with* $h_n=\frac{\tau_{n+1}}{\tau_n}$*, have the form (i.e., to be read as follows: the* $(k-\ell)$*th subdiagonal is given by the diagonal matrix in front of* $\Lambda^{k-\ell}$*)*

$$L_1^k = \sum_{\ell=0}^{\infty} diag\left(\frac{\mathbf{s}_\ell(\tilde\partial_t)\tau_{n+k-\ell+1}\circ\tau_n}{\tau_{n+k-\ell+1}\tau_n}\right)_{n\in\mathbb{Z}}\Lambda^{k-\ell}$$
$$hL_2^{\top k}h^{-1} = \sum_{\ell=0}^{\infty} diag\left(\frac{\mathbf{s}_\ell(-\tilde\partial_s)\tau_{n+k-\ell+1}\circ\tau_n}{\tau_{n+k-\ell+1}\tau_n}\right)_{n\in\mathbb{Z}}\Lambda^{k-\ell}, \tag{3.3.7}$$

and satisfy the **2d-Toda Lattice**[27]

$$\frac{\partial L_i}{\partial t_n}=[(L_1^n)_+\ ,L_i] \quad \textit{and} \quad \frac{\partial L_i}{\partial s_n}=[(L_2^n)_-\ ,L_i]\ ,\quad i=1,2, \tag{3.3.8}$$

---

[25]For the Hirota symbol, see footnote 14. The $J_k^{(i)}$'s are as in remark 1 at the end of Theorem 2.1, for $\beta=1$ and $\tilde J_k^{(i)}=J_k^{(i)}|_{t\to -s}$, with

$$J_k^{(1)}=\frac{\partial}{\partial t_k}+(-k)t_{-k},\quad J_k^{(2)}=\sum_{i+j=k}\frac{\partial^2}{\partial t_i\partial t_j}+2\sum_{-i+j=k}it_i\frac{\partial}{\partial t_j}+\sum_{-i-j=k}it_ijt_j.$$

[26]in terms of the Wronskian $\{f,g\}_t=\frac{\partial f}{\partial t}g-f\frac{\partial g}{\partial t}$.

[27]P_+ and P_- denote the upper (including diagonal) and strictly lower triangular parts of the matrix P, respectively.

(iv) *Bi-orthogonal polynomials: The expressions*

$$
\begin{aligned}
p_n^{(1)}(t,s;y) &:= (S_1(t,s)\chi)y))_n = y^n \frac{\tau_n(t-[y^{-1}],s)}{\tau_n(t,s)} \\
p_n^{(2)}(t,s;z) &:= \left(hS_2^{\top-1}(t,s)\chi(z)\right)_n = z^n \frac{\tau_n(t,s+[z^{-1}])}{\tau_n(t,s)}
\end{aligned}
\tag{3.3.9}
$$

form a system of monic bi-orthogonal polynomials in z*:*

$$
\langle p_n^{(1)}(t,s;y), p_m^{(2)}(t,s;z)\rangle_{t,s} = \delta_{n,m} h_n \quad \textit{with} \quad h_n = \frac{\tau_{n+1}}{\tau_n}, \tag{3.3.10}
$$

which also are eigenvectors of L_1 *and* L_2*:*

$$
zp_n^{(1)}(t,s;z) = L_1(t,s)p_n^{(1)}(t,s;z) \quad \textit{and} \quad zp_n^{(2)}(t,s;z) = L_2^{\top}(t,s)p_n^{(2)}(t,s;z). \tag{3.3.11}
$$

Remark: Notice that every statement can be dualized, upon using the duality $t \longleftrightarrow -s$, $L_1 \longleftrightarrow hL_2^{\top}h^{-1}$.

3.3.2 Sketch of proof

Identity (3.3.4) follows from the fact that the product of the two Vandermonde appearing in the integral (3.3.4) can be expressed as sum of determinants:

$$
\Delta_n(u)\Delta_n(v) = \sum_{\sigma\in S_n} \det\left(u_{\sigma(k)}^{\ell-1} v_{\sigma(k)}^{k-1}\right)_{1\le \ell,k\le n}, \tag{3.3.12}
$$

together with the Harish-Chandra, Itzykson and Zuber formula [34, 38]

$$
\int_{U(n)} dU\, e^{c\,\mathrm{Tr}\, xUy\bar{U}^{\top}} = \frac{(2\pi)^{\frac{n(n-1)}{2}}}{n!} \frac{\det(e^{cx_i y_j})_{1\le i,j\le n}}{\Delta_n(x)\Delta_n(y)}. \tag{3.3.13}
$$

Moreover the τ_n's satisfy the following bilinear identities, for all integer $m, n \ge 0$ and $t, s \in \mathbb{C}^{\infty}$:

$$
\oint_{z=\infty} \tau_n(t-[z^{-1}],s)\tau_{m+1}(t'+[z^{-1}],s')e^{\sum_1^\infty (t_i-t_i')z^i} z^{n-m-1} dz
$$

$$
= \oint_{z=0} \tau_{n+1}(t,s-[z])\tau_m(t',s'+[z])e^{\sum_1^\infty (s_i-s_i')z^{-i}} z^{n-m-1} dz. \tag{3.3.14}
$$

Again, the bi-orthogonal nature (3.3.10) of the polynomials (3.3.9) is tantamount to the Borel decomposition, written in the form $S_1 m_\infty (hS_2^{\top-1})^{\top} = h$. These polynomials

satisfy the eigenvalue problem (3.3.11) and evolve in t, s according to the differential equations

$$\frac{\partial p^{(1)}}{\partial t_n} = -(L_1^n)_- p^{(1)} \qquad \frac{\partial p^{(1)}}{\partial s_n} = -(L_2^n)_- p^{(1)}$$
$$\frac{\partial p^{(2)}}{\partial t_n} = -\left((h^{-1}L_1 h)^{\top n}\right)_- p^{(2)} \qquad \frac{\partial p^{(2)}}{\partial s_n} = \left((h^{-1}L_2 h)^{\top n}\right)_- p^{(2)}. \tag{3.3.15}$$

From the representation (3.3.7) and from the bilinear identity (3.3.14), it follows that

$$\frac{p_{k-1}(\tilde{\partial}_t)\tau_{n+2}\circ\tau_n}{\tau_{n+1}^2} = -\frac{\partial^2}{\partial s_1 \partial t_k}\log\tau_{n+1}, \tag{3.3.16}$$

and so, for $k=1$,

$$\frac{\tau_n\tau_{n+2}}{\tau_{n+1}^2} = -\frac{\partial^2}{\partial s_1\partial t_1}\log\tau_{n+1}. \tag{3.3.17}$$

Thus, using (3.3.7), (3.3.16) and (3.3.17), we have

$$\begin{aligned} \left(L_1^k\right)_{n,n+1} &= \frac{p_{k-1}(\tilde{\partial}_t)\tau_{n+2}\circ\tau_n}{\tau_{n+2}\tau_n} = \frac{\frac{\partial^2\log\tau_{n+1}}{\partial s_1\partial t_k}}{\frac{\partial^2\log\tau_{n+1}}{\partial s_1\partial t_1}} \\ \left(hL_2^{\top k}h^{-1}\right)_{n,n+1} &= \frac{p_{k-1}(-\tilde{\partial}_s)\tau_{n+2}\circ\tau_n}{\tau_{n+2}\tau_n} = \frac{\frac{\partial^2\log\tau_{n+1}}{\partial t_1\partial s_k}}{\frac{\partial^2\log\tau_{n+1}}{\partial s_1\partial t_1}}. \end{aligned} \tag{3.3.18}$$

Combining (3.3.18) with (3.3.17) for $k=2$ yields

$$\begin{aligned} \left(L_1^2\right)_{n,n+1} = \frac{\frac{\partial^2}{\partial s_1\partial t_2}\log\tau_{n+1}}{\frac{\partial^2}{\partial s_1\partial t_1}\log\tau_{n+1}} &= \frac{\partial}{\partial t_1}\log\left(-\frac{\tau_{n+2}}{\tau_n}\right) \\ &= \frac{\partial}{\partial t_1}\log\left(\left(\frac{\tau_{n+1}}{\tau_n}\right)^2\frac{\partial^2}{\partial s_1\partial t_1}\log\tau_{n+1}\right). \end{aligned} \tag{3.3.19}$$

Then, subtracting $\partial/\partial s_1$ of (3.3.19) from $\partial/\partial t_1$ of the dual of the same equation (see remark at the end of Theorem 3.4) leads to (3.3.6). ■

3.4 The Toeplitz Lattice and Unitary matrix integrals

3.4.1 Toeplitz lattice, factorization of moment matrices and bi-orthogonal polynomials

Consider the inner-product

$$\langle f(z), g(z)\rangle_{t,s} := \oint_{S^1}\frac{dz}{2\pi i z}f(z)g(z^{-1})e^{\sum_1^\infty(t_i z^i - s_i z^{-i})}, \qquad t,s\in\mathbb{C}^\infty, \tag{3.4.1}$$

where the integral is taken over the unit circle $S^1 \subset \mathbb{C}$ around the origin. It has the property

$$\langle z^k f, g\rangle_{t,s} = \langle f, z^{-k} g\rangle_{t,s}. \tag{3.4.2}$$

The t, s dependent semi-infinite moment matrix $m_\infty(t,s)$, where[28]

$$\begin{aligned} m_n(t,s) := \left(\langle z^k, z^\ell\rangle_{t,s}\right)_{0\le k,\ell\le n-1} &= \left(\oint_{S^1} \frac{\rho(z)dz}{2\pi i z} z^{k-\ell} e^{\sum_1^\infty (t_i z^i - s_i z^{-i})}\right)_{0\le k,\ell\le n-1} \\ &= \text{Toeplitz matrix} \end{aligned} \tag{3.4.3}$$

satisfies the same differential equations, as in (3.3.3):

$$\frac{\partial m_\infty}{\partial t_n} = \Lambda^n m_\infty \quad \text{and} \quad \frac{\partial m_\infty}{\partial s_n} = -m_\infty \Lambda^{\top n}. \qquad \textit{(2-Toda Lattice)} \tag{3.4.4}$$

As before, define

$$\tau_n(t,s) := \det m_n(t,s).$$

Also, consider the factorization $m_\infty(t,s) = S_1^{-1}(t,s) S_2(t,s)$, as in (3.3.2), from which one defines $L_1 := S_1 \Lambda S_1^{-1}$ and $L_2 := S_2 \Lambda^\top S_2^{-1}$ and the bi-orthogonal polynomials $p_i^{(k)}(t,s;z)$ for $k = 1,2$. Since m_∞ satisfies the same equations (3.3.3), the matrices L_1 and L_2 satisfy the 2-Toda lattice equations; the Toeplitz nature of m_∞ implies a peculiar "rank 2"-structure, with $\frac{h_i}{h_{i-1}} = 1 - x_i y_i$ and $x_0 = y_0 = 1$:

$$h^{-1} L_1 h = \begin{pmatrix} -x_1 y_0 & 1 - x_1 y_1 & 0 & 0 & \\ -x_2 y_0 & -x_2 y_1 & 1 - x_2 y_2 & 0 & \\ -x_3 y_0 & -x_3 y_1 & -x_3 y_2 & 1 - x_3 y_3 & \\ -x_4 y_0 & -x_4 y_1 & -x_4 y_2 & -x_4 y_3 & \\ & & & & \ddots \end{pmatrix}$$

and

$$L_2 = \begin{pmatrix} -x_0 y_1 & -x_0 y_2 & -x_0 y_3 & -x_0 y_4 & \\ 1 - x_1 y_1 & -x_1 y_2 & -x_1 y_3 & -x_1 y_4 & \\ 0 & 1 - x_2 y_2 & -x_2 y_3 & -x_2 y_4 & \\ 0 & 0 & 1 - x_3 y_3 & -x_3 y_4 & \\ & & & & \ddots \end{pmatrix}. \tag{3.4.5}$$

Some of the ideas in the next theorem are inspired by the work of Hisakado [37].

[28] A matrix is Toeplitz, when its (i,j)th entry depends on $i - j$.

Theorem 3.5 (Adler-van Moerbeke [7]) *The integrals $I_n(t,s)$, with $I_0=1$,*

$$\begin{aligned}\tau_n(t,s)=\det m_n=\frac{1}{n!}I_n: &= \frac{1}{n!}\int_{(S^1)^n}|\Delta_n(z)|^2\prod_{k=1}^n\left(e^{\sum_1^\infty(t_iz_k^i-s_iz_k^{-i})}\frac{dz_k}{2\pi iz_k}\right)\\ &= \int_{U(n)}e^{\sum_1^\infty Tr(t_iM^i-s_i\bar M^i)}dM\\ &= \sum_{\{Young\ diagrams\ \lambda\ \mid\ \hat\lambda_1\le n\}}\mathbf{s}_\lambda(t)\mathbf{s}_\lambda(-s),\end{aligned}\tag{3.4.6}$$

satisfy:

(i) *a SL(2,$\mathbb{Z}$)-algebra of three Virasoro constraints (2.3.2):*

$$\mathbb{J}^{(2)}_{k,n}(t,n)-\mathbb{J}^{(2)}_{-k,n}(-s,n)-k\left(\theta\mathbb{J}^{(1)}_{k,n}(t,n)+(1-\theta)\mathbb{J}^{(1)}_{-k,n}(-s,n)\right)I_n(t,s)=0,\tag{3.4.7}$$

$$for\begin{cases}k=-1,\ \theta=0\\ k=0,\ \ \theta\ \ arbitrary\\ k=1,\ \theta=1\end{cases}$$

(ii) *2d-Toda identities: The matrices L_1 and L_2, defined above, satisfy the 2-Toda lattice equations (3.3.8); in particular:*

$$\frac{\partial^2}{\partial s_1\partial t_1}\log\tau_n=-\frac{\tau_{n-1}\tau_{n+1}}{\tau_n^2}$$

and

$$\frac{\partial^2}{\partial s_2\partial t_1}\log\tau_n=-2\frac{\partial}{\partial s_1}\log\frac{\tau_n}{\tau_{n-1}}\ .\frac{\partial^2}{\partial s_1\partial t_1}\log\tau_n-\frac{\partial^3}{\partial s_1^2\partial t_1}\log\tau_n,\tag{3.4.8}$$

the first being equivalent to the discrete sinh-Gordon equation

$$\frac{\partial^2 q_n}{\partial t_1\partial s_1}=e^{q_n-q_{n-1}}-e^{q_{n+1}-q_n},\quad where\ \ q_n=\log\frac{\tau_{n+1}}{\tau_n}.$$

(iii) *The Toeplitz lattice : The 2-Toda lattice solution is a very special one, namely the matrices L_1 and L_2 have a "rank 2" structure, given by (3.4.5), whose x_n's and*

y_n's equal[29]*:*

$$
\begin{aligned}
&x_n(t,s)\\
&= \frac{1}{\tau_n}\int_{U(n)} \mathbf{s}_n(-\operatorname{Tr} M, -\frac{1}{2}\operatorname{Tr} M^2, -\frac{1}{3}\operatorname{Tr} M^3, ...)e^{\sum_1^\infty \operatorname{Tr}(t_i M^i - s_i \bar{M}^i)} dM\\
&= \frac{\mathbf{s}_n(-\frac{\partial}{\partial t_1}, -\frac{1}{2}\frac{\partial}{\partial t_2}, -\frac{1}{3}\frac{\partial}{\partial t_3}, ...)\tau_n(t,s)}{\tau_n(t,s)} = p_n^{(1)}(t,s;0)\\
&y_n(t,s)\\
&= \frac{1}{\tau_n}\int_{U(n)} \mathbf{s}_n(-\operatorname{Tr} \bar{M}, -\frac{1}{2}\operatorname{Tr} \bar{M}^2, -\frac{1}{3}\operatorname{Tr} \bar{M}^3, ...)e^{\sum_1^\infty \operatorname{Tr}(t_i M^i - s_i \bar{M}^i)} dM\\
&= \frac{\mathbf{s}_n(\frac{\partial}{\partial t_1}, \frac{1}{2}\frac{\partial}{\partial t_2}, \frac{1}{3}\frac{\partial}{\partial t_3}, ...)\tau_n(t,s)}{\tau_n(t,s)} = p_n^{(2)}(t,s;0),
\end{aligned}
\tag{3.4.9}
$$

and satisfy the following integrable Hamiltonian system

$$
\begin{aligned}
\frac{\partial x_n}{\partial t_i} &= (1 - x_n y_n)\frac{\partial H_i^{(1)}}{\partial y_n} & \frac{\partial y_n}{\partial t_i} &= -(1 - x_n y_n)\frac{\partial H_i^{(1)}}{\partial x_n}\\
\frac{\partial x_n}{\partial s_i} &= (1 - x_n y_n)\frac{\partial H_i^{(2)}}{\partial y_n} & \frac{\partial y_n}{\partial s_i} &= -(1 - x_n y_n)\frac{\partial H_i^{(2)}}{\partial x_n},
\end{aligned}
\tag{3.4.10}
$$

(**Toeplitz lattice**)

with <u>initial condition</u> $x_n(0,0) = y_n(0,0) = 0$ for $n \geq 1$ and <u>boundary</u> <u>condition</u> $x_0(t,s) = y_0(t,s) = 1$. The traces

$$H_i^{(k)} = -\frac{1}{i} Tr\ L_k^i, \quad i = 1,2,3,..., \quad k = 1,2.$$

of the matrices L_i in (3.4.5) are integrals in involution with regard to the symplectic structure $\omega := \sum_0^\infty (1 - x_k y_k)^{-1} dx_k \wedge dy_k$. The Toeplitz nature of m_∞ leads to identities between τ's, the simplest being (due to Hisakado [37]) :

$$\left(1 + \frac{\partial^2}{\partial s_1 \partial t_1}\log \tau_{n+1}\right)\left(1 + \frac{\partial^2}{\partial s_1 \partial t_1}\log \tau_n\right) = -\frac{\partial}{\partial t_1}\log\frac{\tau_{n+1}}{\tau_n}\frac{\partial}{\partial s_1}\log\frac{\tau_{n+1}}{\tau_n}. \tag{3.4.11}$$

<u>*Remark:*</u> The first equation in the hierarchy above reads:

$$
\begin{aligned}
\frac{\partial x_n}{\partial t_1} &= x_{n+1}(1 - x_n y_n) & \frac{\partial y_n}{\partial t_1} &= -y_{n-1}(1 - x_n y_n)\\
\frac{\partial x_n}{\partial s_1} &= x_{n-1}(1 - x_n y_n) & \frac{\partial y_n}{\partial s_1} &= -y_{n+1}(1 - x_n y_n).
\end{aligned}
$$

[29] Remember $\mathbf{s}(t_1, t_2, ...)$ are elementary Schur polynomials.

3.4.2 Sketch of Proof

The identity (3.4.6) between the determinant and the moment matrix uses again the Vandermonde identity (3.3.12),

$$
\begin{aligned}
&\int_{U(n)} e^{\sum_1^\infty Tr(t_i M^i - s_i \bar{M}^i)} dM \\
&\quad = \int_{(S^1)^n} |\Delta_n(z)|^2 \prod_{k=1}^n \left(e^{\sum_1^\infty (t_i z_k^i - s_i z_k^{-i})} \frac{dz_k}{2\pi i z_k} \right) \\
&\quad = \int_{(S^1)^n} \Delta_n(z)\Delta_n(\bar{z}) \prod_{k=1}^n \left(e^{\sum_1^\infty (t_i z_k^i - s_i z_k^{-i})} \frac{dz_k}{2\pi i z_k} \right) \\
&\quad = \int_{(S^1)^n} \sum_{\sigma\in S_n} \det \left(z_{\sigma(m)}^{\ell-1} \bar{z}_{\sigma(m)}^{m-1} \right)_{1\le \ell,m\le n} \prod_{k=1}^n \left(e^{\sum_1^\infty (t_i z_k^i - s_i z_k^{-i})} \frac{dz_k}{2\pi i z_k} \right) \\
&\quad = \sum_{\sigma\in S_n} \det \left(\oint_{S^1} z_k^{\ell-1} \bar{z}_k^{m-1} e^{\sum_1^\infty (t_i z_k^i - s_i z_k^{-i})} \frac{dz_k}{2\pi i z_k} \right)_{1\le \ell,m\le n} \\
&\quad = n! \det \left(\oint_{S^1} z^{\ell-m} e^{\sum_1^\infty (t_i z^i - s_i z^{-i})} \frac{dz}{2\pi i z} \right)_{1\le \ell,m\le n} = n! \det m_n(t,s) = n! \tau_n(t)
\end{aligned}
$$

Using $z^{k\top} = z^{-k}$ (see (3.4.2)), one shows that the polynomials $p_{n+1}^{(1)}(z) - z p_n^{(1)}(z)$ and $p_{n+1}^{(1)}(0) z^n p_n^{(2)}(z^{-1})$ are perpendicular to the monomials $z^0, z^1, ..., z^n$ and that they have the same z^0-term; one makes a similar argument, by dualizing $1 \leftrightarrow 2$. Therefore, we have the Hisakado identities between the following polynomials:

$$
\begin{aligned}
p_{n+1}^{(1)}(z) - z p_n^{(1)}(z) &= p_{n+1}^{(1)}(0) z^n p_n^{(2)}(z^{-1}) \\
p_{n+1}^{(2)}(z) - z p_n^{(2)}(z) &= p_{n+1}^{(2)}(0) z^n p_n^{(1)}(z^{-1}).
\end{aligned} \tag{3.4.12}
$$

The rank 2 structure (3.4.5) of L_1 and L_2, with $x_n = p_n^{(1)}(t,s;0)$ and $y_n = p_n^{(2)}(t,s;0)$, is obtained by taking the inner-product of $p_{n+1}^{(1)}(z) - z p_n^{(1)}(z)$ with itself, for different n and m, and using the fact that $z p_n^{(1)}(z) = L_1 p_n^{(1)}(z)$.

To check the first equation in the hierarchy (see remark at the end of theorem 3.5),

consider, from (3.4.9),

$$\begin{aligned}
\frac{\partial x_n}{\partial t_1} &= \left.\frac{\partial p_n^{(1)}(t,s;z)}{\partial t_1}\right|_{z=0} \\
&= -\left((L_1)_- p^{(1)}\right)_n\Big|_{z=0}, \quad \text{using (3.3.15)}, \\
&= h_n p_{n+1}^{(1)}(t,s;0)\sum_{i=0}^{n-1}\frac{p_i^{(1)}(t,s;0)p_i^{(2)}(t,s;0)}{h_i}, \quad \text{using (3.4.5)}, \\
&= h_n x_{n+1}\sum_{i=0}^{n-1}\frac{x_i y_i}{h_i} \\
&= h_n x_{n+1}\sum_{i=0}^{n-1}\left(\frac{1}{h_i}-\frac{1}{h_{i-1}}\right), \quad \text{using } \frac{h_i}{h_{i-1}} = 1 - x_i y_i, \\
&= x_{n+1}\frac{h_n}{h_{n-1}} \\
&= x_{n+1}(1 - x_n y_n),
\end{aligned}$$

and similarly for the other coordinates. From (3.3.7) and (3.4.5), upon making the products of the corresponding diagonal entries of L_1 and $hL_2^\top h^{-1}$, one finds (3.4.11):

$$\begin{aligned}
\frac{\partial}{\partial t_1}\log\frac{\tau_{n+1}}{\tau_n}\frac{\partial}{\partial s_1}\log\frac{\tau_{n+1}}{\tau_n} = -x_{n+1}y_n x_n y_{n+1} &= -x_n y_n x_{n+1} y_{n+1} \\
&= -\left(1 - \frac{h_n}{h_{n-1}}\right)\left(1 - \frac{h_{n+1}}{h_n}\right).
\end{aligned}$$

■

4 Ensembles of finite random matrices

4.1 PDE's defined by the probabilities in Hermitian, symmetric and symplectic random ensembles

As used earlier, the disjoint union $E = \cup_1^{2r}[c_{2i-1}, c_{2i}] \subset \mathbb{R}$, and the weight $\rho(z) = e^{-V(z)}$, with $-\rho'/\rho = V' = g/f$ define an algebra of differential operators, $(k \in \mathbb{Z})$

$$\mathcal{B}_k = \sum_1^{2r} c_i^k f(c_i)\frac{\partial}{\partial c_i}.$$

The aim of this section is to find PDE's for the following probabilities in terms of the boundary points c_i of E (see (1.1.9), (1.1.11) and (1.1.18)), i.e.

$$\begin{aligned} P_n(E) : &= P_n(\text{ all spectral points of } M \in E) \\ &= \frac{\int_{\mathcal{H}_n(E),\ \mathcal{S}_n(E) \text{ or } \mathcal{T}_n(E)} e^{-tr\ V(M)} dM}{\int_{\mathcal{H}_n(\mathbb{R}),\ \mathcal{S}_n(\mathbb{R}) \text{ or } \mathcal{T}_n(\mathbb{R})} e^{-tr\ V(M)} dM} \\ &= \frac{\int_{E^n} |\Delta_n(z)|^\beta \prod_{k=1}^n e^{-V(z_k)} dz_k}{\int_{\mathbb{R}^n} |\Delta_n(z)|^\beta \prod_{k=1}^n e^{-V(z_k)} dz_k} \qquad \beta = 2, 1, 4 \text{ respectively,} \end{aligned} \tag{4.1.1}$$

involving the classical weights below. In anticipation, the equations obtained in Theorems 4.1, 4.2 and 4.3 are closely related to three of the six Painlevé differential equations:

weight	$\rho(z)$	Painlevé
Gauss	e^{-bz^2}	IV
Laguerre	$z^a e^{-bz}$	V
Jacobi	$(1-z)^a(1+z)^b$	VI

For $\beta = 2$, the probabilities satisfy partial differential equations in the boundary points of E, whereas in the case $\beta = 1, 4$, the equations are inductive. Namely, for $\beta = 1$ (resp. $\beta = 4$) , the probabilities P_{n+2} (resp. P_{n+1}) are given in tems of P_{n-2} (resp. P_{n-1}) and a differential operator acting on P_n. The weights above involve the parameters β, a, b and

$$\delta^\beta_{1,4} := 2\left(\left(\frac{\beta}{2}\right)^{1/2} - \left(\frac{\beta}{2}\right)^{-1/2}\right)^2 = \begin{cases} 0 & \text{for } \beta = 2 \\ 1 & \text{for } \beta = 1, 4. \end{cases}$$

As a consequence of the duality (2.1.12) between β-Virasoro generators under the map $\beta \mapsto 4/\beta$, and the equations (2.1.7), the PDE's obtained have a remarkable property: the coefficients Q and Q_i of the PDE's are functions in the variables n, β, a, b, having the invariance property under the map

$$n \to -2n,\ a \to -\frac{a}{2},\ b \to -\frac{b}{2};$$

to be precise,

$$Q_i(-2n, \beta, -\frac{a}{2}, -\frac{b}{2})\Big|_{\beta=1} = Q_i(n, \beta, a, b)|_{\beta=4}\,. \tag{4.1.2}$$

The results in this section are mainly due to Adler-Shiota-van Moerbeke [11] for $\beta = 2$ and Adler-van Moerbeke [6] for $\beta = 1, 4$. For more detailed references, see the end of section 4.2.

4.1.1 Gaussian Hermitian, symmetric and symplectic ensembles

Given the disjoint union E and the weight e^{-bz^2}, the differential operators $\mathcal{B}_k$ take on the form

$$\mathcal{B}_k = \sum_1^{2r} c_i^{k+1} \frac{\partial}{\partial c_i}.$$

Define the *invariant* polynomials

$$Q = 12b^2 n \left(n + 1 - \frac{2}{\beta} \right) \qquad \text{and} \qquad Q_2 = 4(1 + \delta_{1,4}^{\beta}) b \left(2n + \delta_{1,4}^{\beta} (1 - \frac{2}{\beta}) \right).$$

Theorem 4.1 *The following probabilities ($\beta = 2, 1, 4$)*

$$P_n(E) = \frac{\int_{E^n} |\Delta_n(z)|^{\beta} \prod_{k=1}^n e^{-bz_k^2} dz_k}{\int_{\mathbb{R}^n} |\Delta_n(z)|^{\beta} \prod_{k=1}^n e^{-bz_k^2} dz_k}, \tag{4.1.3}$$

satisfy the PDE's ($F := F_n = \log P_n$):

$$\delta_{1,4}^{\beta} Q \left(\frac{P_{n-\frac{2}{1}} P_{n+\frac{2}{1}}}{P_n^2} - 1 \right) \qquad \textit{with index} \begin{cases} 2 & \textit{when } n \textit{ even and } \beta = 1 \\ 1 & \textit{when } n \textit{ arbitrary and } \beta = 4 \end{cases}$$

$$= \left(\mathcal{B}_{-1}^4 + (Q_2 + 6\mathcal{B}_{-1}^2 F) \mathcal{B}_{-1}^2 + 4(2 - \delta_{1,4}^{\beta}) \frac{b^2}{\beta} (3\mathcal{B}_0^2 - 4\mathcal{B}_{-1}\mathcal{B}_1 + 6\mathcal{B}_0) \right) F. \tag{4.1.4}$$

4.1.2 Laguerre Hermitian, symmetric and symplectic ensembles

Given the disjoint union $E \subset \mathbb{R}^+$ and the weight $z^a e^{-bz}$, the $\mathcal{B}_k$ take on the form

$$\mathcal{B}_k = \sum_1^{2r} c_i^{k+2} \frac{\partial}{\partial c_i}.$$

Define the polynomials, also respecting the duality (4.1.2),

$$\begin{aligned}
Q &= \begin{cases} \frac{3}{4} n(n-1)(n+2a)(n+2a+1), & \text{for } \beta = 1 \\[2ex] \frac{3}{2} n(2n+1)(2n+a)(2n+a-1), & \text{for } \beta = 4 \end{cases} \\
Q_2 &= \left(3\beta n^2 - \frac{a^2}{\beta} + 6an + 4(1 - \frac{\beta}{2}) a + 3 \right) \delta_{1,4}^{\beta} + (1 - a^2)(1 - \delta_{1,4}^{\beta}) \\
Q_1 &= \left(\beta n^2 + 2an + (1 - \frac{\beta}{2}) a \right), \qquad Q_0 = b(2 - \delta_{1,4}^{\beta})(n + \frac{a}{\beta}).
\end{aligned}$$

Theorem 4.2 *The following probabilities*

$$P_n(E) = \frac{\int_{E^n} |\Delta_n(z)|^\beta \prod_{k=1}^n z_k^a e^{-bz_k} dz_k}{\int_{\mathbb{R}_+^n} |\Delta_n(z)|^\beta \prod_{k=1}^n z_k^a e^{-bz_k} dz_k} \tag{4.1.5}$$

satisfy the PDE [30] *: ($F := F_n = \log P_n$)*

$$\delta_{1,4}^\beta Q \left(\frac{P_{n-{2 \atop 1}} P_{n+{2 \atop 1}}}{P_n^2} - 1 \right)$$

$$= \Big(\mathcal{B}_{-1}^4 - 2(\delta_{1,4}^\beta + 1)\mathcal{B}_{-1}^3 + (Q_2 + 6\mathcal{B}_{-1}^2 F - 4(\delta_{1,4}^\beta + 1)\mathcal{B}_{-1}F)\mathcal{B}_{-1}^2 - 3\delta_{1,4}^\beta (Q_1 - \mathcal{B}_{-1}F)\mathcal{B}_{-1}$$
$$+ \frac{b^2}{\beta}(2 - \delta_{1,4}^\beta)(3\mathcal{B}_0^2 - 4\mathcal{B}_1\mathcal{B}_{-1} - 2\mathcal{B}_1) + Q_0(2\mathcal{B}_0\mathcal{B}_{-1} - \mathcal{B}_0) \Big) F. \tag{4.1.6}$$

4.1.3 Jacobi Hermitian, symmetric and symplectic ensembles

In terms of $E \subset [-1, +1]$ and the Jacobi weight $(1-z)^a(1+z)^b$, the differential operators $\mathcal{B}_k$ take on the form

$$\mathcal{B}_k = \sum_1^{2r} c_i^{k+1}(1 - c_i^2)\frac{\partial}{\partial c_i}.$$

Introduce the following variables, which themselves have the invariance property (4.1.2) ($b_0 = a - b,\ b_1 = a + b$):

$$r = \frac{4}{\beta}(b_0^2 + (b_1 + 2 - \beta)^2) \qquad s = \frac{4}{\beta} b_0 (b_1 + 2 - \beta)$$

$$q_n = \frac{4}{\beta}(\beta n + b_1 + 2 - \beta)(\beta n + b_1),$$

and the following *invariant* polynomials in q, r, s:

$$\begin{aligned}
Q &= \frac{3}{16}\left((s^2 - qr + q^2)^2 - 4(rs^2 - 4qs^2 - 4s^2 + q^2 r)\right) \\
Q_1 &= 3s^2 - 3qr - 6r + 2q^2 + 23q + 24 \\
Q_2 &= 3qs^2 + 9s^2 - 4q^2 r + 2qr + 4q^3 + 10q^2 \\
Q_3 &= 3qs^2 + 6s^2 - 3q^2 r + q^3 + 4q^2 \\
Q_4 &= 9s^2 - 3qr - 6r + q^2 + 22q + 24 = Q_1 + (6s^2 - q^2 - q).
\end{aligned} \tag{4.1.7}$$

Theorem 4.3 *The following probabilities*

$$P_n(E) = \frac{\int_{E^n} |\Delta_n(z)|^\beta \prod_{k=1}^n (1 - z_k)^a (1 + z_k)^b dz_k}{\int_{[-1,1]^n} |\Delta_n(z)|^\beta \prod_{k=1}^n (1 - z_k)^a (1 + z_k)^b dz_k} \tag{4.1.8}$$

satisfy the PDE ($F = F_n = \log P_n$):

[30] with the same convention on the indices $n \pm 2$ and $n \pm 1$, as in (4.1.4)

<u>*for $\beta = 2$:*</u>

$$\begin{aligned}\Big(2\mathcal{B}_{-1}^4 + (q-r+4)\mathcal{B}_{-1}^2 - (4\mathcal{B}_{-1}F - s)\mathcal{B}_{-1} + 3q\mathcal{B}_0^2 - 2q\mathcal{B}_0 + 8\mathcal{B}_0\mathcal{B}_{-1}^2 \\ -4(q-1)\mathcal{B}_1\mathcal{B}_{-1} + (4\mathcal{B}_{-1}F - s)\mathcal{B}_1 + 2(4\mathcal{B}_{-1}F - s)\mathcal{B}_0\mathcal{B}_{-1} + 2q\mathcal{B}_2\Big)F \\ +4\mathcal{B}_{-1}^2F\left(2\mathcal{B}_0F + 3\mathcal{B}_{-1}^2F\right) = 0 \end{aligned} \tag{4.1.9}$$

<u>*for $\beta = 1,4$:*</u>

$$\begin{aligned} &Q\left(\frac{P_{n+{2\atop 1}}P_{n-{2\atop 1}}}{P_n^2} - 1\right) \\ &\quad = (q+1)\Big(4q\mathcal{B}_{-1}^4 + 12(4\mathcal{B}_{-1}F - s)\mathcal{B}_{-1}^3 + 2\,(q+12)\,(4\mathcal{B}_{-1}F - s)\mathcal{B}_0\mathcal{B}_{-1} \\ &\quad +3q^2\mathcal{B}_0^2 - 4\,(q-4)\,q\mathcal{B}_1\mathcal{B}_{-1} + q(4\mathcal{B}_{-1}F - s)\mathcal{B}_1 + 20q\mathcal{B}_0\mathcal{B}_{-1}^2 + 2q^2\mathcal{B}_2\Big)F \\ &\quad +\Big(Q_2\mathcal{B}_{-1}^2 - sQ_1\mathcal{B}_{-1} + Q_3\mathcal{B}_0\Big)F + 48(\mathcal{B}_{-1}F)^4 - 48s(\mathcal{B}_{-1}F)^3 + 2Q_4(\mathcal{B}_{-1}F)^2 \\ &\quad +12\,q^2(\mathcal{B}_0F)^2 + 16\,q\,\left(2\,q-1\right)(\mathcal{B}_{-1}^2F)(\mathcal{B}_0F) + 24\,(q-1)\,q(\mathcal{B}_{-1}^2F)^2 \\ &\quad +24\Big(2\mathcal{B}_{-1}F - s\Big)\Big((q+2)\mathcal{B}_0F + (q+3)\mathcal{B}_{-1}^2F\Big)\mathcal{B}_{-1}F. \end{aligned} \tag{4.1.10}$$

The *Proof* of these three theorems will be sketched in subsections 4.3, 4.4 and 4.5.

4.2 ODE's, when E has one boundary point

Assume the set E consists of one boundary point $c = x$, besides the boundary of the full range; thus, setting respectively $E = [-\infty, x], E = [0, x], E = [-1, x]$ in the PDE's (4.1.4), (4.1.6) and (4.1.9), (4.1.10), leads to the equations in x below. Notice that, for $\beta = 2$, the equations obtained are ODE's and, for $\beta = 1,4$, these equations express P_{n+2} in terms of P_{n-2} and a differential operator acting on P_n:

(1) *Gauss ensemble* $(\beta = 2,1,4)$: $f_n(x) = \frac{d}{dx}\log P_n(\max_i\ \lambda_i \le x)$ satisfies

$$\begin{aligned} &\delta_{1,4}^{\beta}Q\left(\frac{P_{n-{2\atop 1}}P_{n+{2\atop 1}}}{P_n^2} - 1\right) \\ &= \quad f_n''' + 6f_n'^2 + \left(4\frac{b^2x^2}{\beta}(\delta_{1,4}^{\beta} - 2) + Q_2\right)f_n' - 4\frac{b^2x}{\beta}(\delta_{1,4}^{\beta} - 2)f_n. \end{aligned} \tag{4.2.1}$$

(2) *Laguerre ensemble* $(\beta = 2,1,4)$: $f_n(x) = x\frac{d}{dx}\log P_n(\max_i\ \lambda_i \le x)$ (with all eigen-

values $\lambda_i \geq 0$) satisfies:

$$
\begin{aligned}
&\delta^{\beta}_{1,4} Q \left(\frac{P_{n-\frac{2}{1}} P_{n+\frac{2}{1}}}{P_n^2} - 1 \right) - \left(3\delta^{\beta}_{1,4} f_n - \frac{b^2 x^2}{\beta} (\delta^{\beta}_{1,4} - 2) - Q_0 x - 3\delta^{\beta}_{1,4} Q_1 \right) f_n \\
&= x^3 f_n''' - (2\delta^{\beta}_{1,4} - 1) x^2 f_n'' + 6x^2 f_n'^2 \\
&\quad - x \left(4(\delta^{\beta}_{1,4} + 1) f_n - \frac{b^2 x^2}{\beta} (\delta^{\beta}_{1,4} - 2) - 2Q_0 x - Q_2 + 2\delta^{\beta}_{1,4} + 1 \right) f_n'.
\end{aligned} \tag{4.2.2}
$$

(3) *Jacobi ensemble*: $f := f_n(x) = (1 - x^2) \frac{d}{dx} \log P_n(\max_i \lambda_i \leq x)$ (with all eigenvalues $-1 \leq \lambda_i \leq 1$) satisfies:

- for $\underline{\beta = 2}$:

$$
\begin{aligned}
&2(x^2-1)^2 f''' + 4(x^2-1)\left(xf'' - 3f'^2\right) + \left(16xf - q(x^2-1) - 2sx - r\right) f' \\
&\qquad - f\,(4f - qx - s) = 0,
\end{aligned} \tag{4.2.3}
$$

- for $\underline{\beta = 1, 4}$:

$$
\begin{aligned}
&Q \left(\frac{P_{n+\frac{2}{1}} P_{n-\frac{2}{1}}}{P_n^2} - 1 \right) \\
&\quad = 4(q+1)(x^2-1)^2 \Big(-q(x^2-1) f''' + (12f - qx - 3s) f'' + 6q(q-1) f'^2 \Big) \\
&\qquad -(x^2-1) f' \Big(24 f (q+3)(2f - s) + 8 f q (5q-1) x - q(q+1)(qx^2 + 2sx + 8) + Q_2 \Big) \\
&\qquad + f \Big(48 f^3 + 48 f^2 (qx + 2x - s) + 2f \left(8q^2 x^2 + 2qx^2 - 12qsx - 24sx + Q_4 \right) \\
&\qquad\quad - q(q+1) x (3qx^2 + sx - 2qx - 3q) + Q_3 x - Q_1 s \Big).
\end{aligned} \tag{4.2.4}
$$

For $\beta = 2$, the term containing the ratio $\frac{P_{n+\frac{2}{1}} P_{n-\frac{2}{1}}}{P_n^2} - 1$ on the left hand side of (4.2.1), (4.2.2) and (4.2.4) vanishes and one thus obtains the ODE's:

- **Gauss**: $f_n(x) := \frac{d}{dx} \log P_n(\max_i \lambda_i \leq x)$ satisfies:

$$
f''' + 6\ f'^2 + 4b(2n - bx^2) f' + 4b^2 x\ f = 0
$$

- **Laguerre**: $f_n(x) := x \frac{d}{dx} \log P_n(\max_i \lambda_i \leq x)$ satisfies

$$
x^2 f''' + x f'' + 6x f'^2 - 4 f f' - ((a - bx)^2 - 4nbx) f' - b(2n + a - bx) f = 0.
$$

- **Jacobi**: $f_n(x) = (1-x^2)\frac{d}{dx}\log P_n(\max_i \lambda_i \leq x)$ satisfies:

$$2(x^2-1)^2 f''' + 4(x^2-1)\left(xf'' - 3f'^2\right) + \left(16xf - q(x^2-1) - 2sx - r\right)f'$$
$$-f\,(4f - qx - s) = 0.$$

Each of these three equations is of the Chazy form (see section 9)

$$f''' + \frac{P'}{P}f'' + \frac{6}{P}f'^2 - \frac{4P'}{P^2}ff' + \frac{P''}{P^2}f^2 + \frac{4Q}{P^2}f' - \frac{2Q'}{P^2}f + \frac{2R}{P^2} = 0, \qquad (4.2.5)$$

with $c = 0$ and P, Q, R having the form:

Gauss	$P(x) = 1$	$4Q(x) = -4b^2x^2 + 8bn$	$R = 0$
Laguerre	$P(x) = x$	$4Q(x) = -(bx-a)^2 + 4bnx$	$R = 0$
Jacobi	$P(x) = 1 - x^2$	$4Q(x) = -\frac{1}{2}(q(x^2-1) + 2sx + r)$	$R = 0$

Cosgrove shows such a third order equation (4.2.5) in $f(x)$, with $P(x)$, $Q(x)$, $R(x)$ of respective degrees $3, 2, 1$, has a first integral (9.0.2), which is second order in f and quadratic in f'', with an integration constant c. Equation (9.0.2) is a master Painlevé equation, containing the 6 Painlevé equations. If $f(x)$ satisfies the equations above, then the new (renormalized) function $g(z)$, defined below,

Gauss	$g(z) = b^{-1/2}f(zb^{-1/2}) + \frac{2}{3}nz$
Laguerre	$g(z) = f(z) + \frac{b}{4}(2n+a)z + \frac{a^2}{4}$
Jacobi	$g(z) := -\frac{1}{2}f(x)\vert_{x=2z-1} - \frac{q}{8}z + \frac{q+s}{16}$

satisfies the canonical equations, which then can be transformed into the standard Painlevé equations; these canonical equations are respectively:

- $g''^2 = -4g'^3 + 4(zg'-g)^2 + A_1g' + A_2$ (**Painlevé IV**)

- $(zg'')^2 = (zg'-g)\Big(-4g'^2 + A_1(zg'-g) + A_2\Big) + A_3g' + A_4$

  (**Painlevé V**)

- $(z(z-1)g'')^2 = (zg'-g)\Big(4g'^2 - 4g'(zg'-g) + A_2\Big) + A_1g'^2 + A_3g' + A_4$

with respective coefficients (**Painlevé VI**)

- $A_1 = 3\left(\frac{4n}{3}\right)^2,\ A_2 = -\left(\frac{4n}{3}\right)^3$

- $A_1 = b^2,\ A_2 = b^2((n+\frac{a}{2})^2 + \frac{a^2}{2}),\ A_3 = -a^2b(n+\frac{a}{2}),\ A_4 = \frac{(ab)^2}{2}$
  $.((n+\frac{a}{2})^2 + \frac{a^2}{8})$

- $A_1 = \frac{2q+r}{8}$, $A_2 = \frac{qs}{16}$, $A_3 = \frac{(q-s)^2+2qr}{64}$, $A_4 = \frac{q}{512}(2s^2+qr)$.

For $\beta = 1$ and 4, the inductive partial differential equations (4.1.4), (4.1.6), (4.1.10), and the derived differential equations (4.2.1), (4.2.2) and (4.2.4) are due to Adler-van Moerbeke [6]. For $\beta = 2$ and for general E, they were first computed by Adler-Shiota-van Moerbeke [11], using the method of the present paper. For $\beta = 2$ and for E having one boundary point, the equations obtained here coincide with the ones first obtained by Tracy-Widom in [64], who recognized them to be Painlevé IV and V for the Gaussian and Laguerre distribution respectively. In his Louvain doctoral dissertation, J.P. Semengue, together with L. Haine [32], were lead to Painlevé VI for the Jacobi ensemble, for $\beta = 2$ and E having one boundary point, upon subtracting the Tracy-Widom differential equation ([64]) from the one computed with the Adler-Shiota-van Moerbeke method ([11]). Cosgrove's ([23]) and Cosgrove-Scoufis's classification ([24], (A.3),) leads directly to these results.

4.3 Proof of Theorems 4.1, 4.2 and 4.3

4.3.1 Gaussian and Laguerre ensembles

The three first Virasoro equations, as in (2.1.29) and (2.1.32), are differential equations, involving partials in $t \in \mathbb{C}^\infty$ and partials $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3$ in $c = (c_1, ..., c_{2r}) \in \mathbb{R}^{2r}$, for $F := F_n(t,c) = \log I_n$; they have the general form:

$$\mathcal{D}_k F = \frac{\partial F}{\partial t_k} + \sum_{-1 \le j < k} \gamma_{kj} V_j(F) + \gamma_k + \delta_k t_1, \quad k = 1,2,3, \tag{4.3.1}$$

with first $V_j(F)$'s given by:

$$V_j(F) = \sum_{i, i+j \ge 1} i t_i \frac{\partial F}{\partial t_{i+j}} + \frac{\beta}{2}\delta_{2,j}\left(\frac{\partial^2 F}{\partial t_1^2} + \left(\frac{\partial F}{\partial t_1}\right)^2\right), \quad -1 \le j \le 2. \tag{4.3.2}$$

In (4.3.1) and (4.3.2), $\beta > 0, \gamma_{kj}, \gamma_k, \delta_k$ are arbitrary parameters; also $\delta_{2j} = 0$ for $j \ne 2$ and $= 1$ for $j = 2$. The claim is that the equations (4.3.1) enable one to express all partial derivatives,

$$\left.\frac{\partial^{i_1+...+i_k} F(t,c)}{\partial t_1^{i_1} ... \partial t_k^{i_k}}\right|_{\mathcal{L}}, \text{ along } \mathcal{L} := \{\text{all } t_i = 0,\ c = (c_1, ..., c_{2r}) \text{ arbitrary}\}, \tag{4.3.3}$$

uniquely in terms of polynomials in

$$\mathcal{D}_{j_1} ... \mathcal{D}_{j_r} F(0,c).$$

Indeed, the method consists of expressing $\partial F/\partial t_k\big|_{t=0}$ in terms of $\mathcal{D}_k f\big|_{t=0}$, using (4.3.1). Second derivatives are obtained by acting on $\mathcal{D}_k F$ with $\mathcal{D}_\ell$, by noting that $\mathcal{D}_\ell$ commutes

with all t-derivatives, by using the equation for $\mathcal{D}_\ell F$, and by setting in the end $t=0$:

$$\begin{aligned}
\mathcal{D}_\ell\mathcal{D}_k F &= \mathcal{D}_\ell\frac{\partial F}{\partial t_k}+\sum_{-1\leq j<k}\gamma_{kj}\mathcal{D}_\ell(V_j(F))\\
&= \left(\frac{\partial}{\partial t_k}+\sum_{-1\leq j<k}\gamma_{kj}V_j\right)\mathcal{D}_\ell(F), \quad \text{provided } V_j(F) \text{ does not}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{contain non-linear terms}\\
&= \left(\frac{\partial}{\partial t_k}+\sum_{-1\leq j<k}\gamma_{kj}V_j\right)\left(\frac{\partial F}{\partial t_\ell}+\sum_{-1\leq j<\ell}\gamma_{\ell j}V_j(F)+\delta_\ell t_1\right)\\
&= \frac{\partial^2 F}{\partial t_k\partial t_\ell}+\text{lower-weight terms.}
\end{aligned}$$

When the non-linear term is present, it is taken care as follows:

$$\mathcal{D}_\ell\left(\frac{\partial F}{\partial t_1}\right)^2=2\frac{\partial F}{\partial t_1}\mathcal{D}_\ell\frac{\partial F}{\partial t_1} \;=\; 2\frac{\partial F}{\partial t_1}\frac{\partial}{\partial t_1}\left(\frac{\partial F}{\partial t_\ell}+\sum_{-1\leq j<\ell}\gamma_{\ell j}V_j(F)+\gamma_\ell+\delta_\ell t_1\right).$$

Higher derivatives are obtained in the same way. We only record here, for future use, the few partials appearing in the KP equation (3.1.6):

$$\begin{aligned}
\left.\frac{\partial^2 F}{\partial t_1^2}\right|_{\mathcal{L}} &= \left(\mathcal{D}_1^2-\gamma_{10}\mathcal{D}_1\right)F+\gamma_{10}\gamma_1-\delta_1\\
\left.\frac{\partial^4 F}{\partial t_1^4}\right|_{\mathcal{L}} &= \left(\mathcal{D}_1^4-6\gamma_{10}\mathcal{D}_1^3+11\gamma_{10}^2\mathcal{D}_1^2-6\gamma_{10}^3\mathcal{D}_1\right)F-6\gamma_{10}^2(\delta_1-\gamma_1\gamma_{10})\\
\left.\frac{\partial^2 F}{\partial t_2^2}\right|_{\mathcal{L}} &= \Big(\mathcal{D}_2^2-2\gamma_{20}\mathcal{D}_2+\beta\gamma_{21}\gamma_{32}\mathcal{D}_1^2-((2\gamma_1+\gamma_{10})\gamma_{21}\gamma_{32}\beta+2\gamma_{2,-1})\mathcal{D}_1\\
&\quad -2\gamma_{21}\mathcal{D}_3\Big)F+\beta\gamma_{21}\gamma_{32}(\mathcal{D}_1F)^2\\
&\quad +\beta\gamma_{21}\gamma_{32}(\gamma_1^2+\gamma_{10}\gamma_1-\delta_1)+2(\gamma_{21}\gamma_3+\gamma_{20}\gamma_2+\gamma_1\gamma_{2,-1})\\
\left.\frac{\partial^2 F}{\partial t_1\partial t_3}\right|_{\mathcal{L}} &= \Big(\mathcal{D}_1\mathcal{D}_3-\frac{\beta}{2}\gamma_{32}\mathcal{D}_1^3+\beta\gamma_{32}(\gamma_1+2\gamma_{10})\mathcal{D}_1^2-\frac{3\beta}{2}\gamma_{10}\gamma_{32}(2\gamma_1+\gamma_{10})\mathcal{D}_1\\
&\quad -3\gamma_{1,-1}\mathcal{D}_2-3\gamma_{10}\mathcal{D}_3\Big)F+\frac{3\beta}{2}\gamma_{10}\gamma_{32}(\mathcal{D}_1F)^2-\beta\gamma_{32}(\mathcal{D}_1F)(\mathcal{D}_1^2F)\\
&\quad +\frac{3}{2}(2\gamma_{10}\gamma_3+\beta\gamma_{32}\gamma_{10}(\gamma_1^2+\gamma_{10}\gamma_1-\delta_1)+2\gamma_{1,-1}\gamma_2).
\end{aligned}$$

4.3.2 Jacobi ensemble

Here, from the Virasoro constraints (2.1.35), one proceeds in the same way as before, by forming $\mathcal{B}_iF|_{t=0}$, $\mathcal{B}_i\mathcal{B}_jF|_{t=0}$, etc..., in terms of t_i partials. For example, from the expressions $\mathcal{B}_{-1}F|_{t=0}$, $\mathcal{B}_{-1}^2F\big|_{t=0}$, $\mathcal{B}_0F|_{t=0}$, one extracts

$$\left.\frac{\partial F}{\partial t_1}\right|_{t=0},\ \left.\frac{\partial^2 F}{\partial t_1^2}\right|_{t=0},\ \left.\frac{\partial F}{\partial t_2}\right|_{t=0}.$$

From the expressions $\mathcal{B}_{-1}^3 F\big|_{t=0}, \mathcal{B}_0\mathcal{B}_{-1}F\big|_{t=0}, \mathcal{B}_1 F\big|_{t=0}$, and using the previous information, one extracts

$$\left.\frac{\partial F}{\partial t_3}\right|_{t=0}, \left.\frac{\partial^2 F}{\partial t_1^3}\right|_{t=0}, \left.\frac{\partial^2 F}{\partial t_1 \partial t_2}\right|_{t=0}.$$

Finally, from the expressions $\mathcal{B}_2 F\big|_{t=0}, \mathcal{B}_1\mathcal{B}_{-1}F\big|_{t=0}, \mathcal{B}_0^2 F\big|_{t=0}, \mathcal{B}_0\mathcal{B}_{-1}^2 F\big|_{t=0}, \ \mathcal{B}_{-1}^4 F\big|_{t=0}$, one deduces

$$\left.\frac{\partial^4 F}{\partial t_1^4}\right|_{t=0}, \left.\frac{\partial F}{\partial t_4}\right|_{t=0}, \left.\frac{\partial^3 F}{\partial t_1^2 \partial t_2}\right|_{t=0}, \left.\frac{\partial^2 F}{\partial t_1 \partial t_3}\right|_{t=0}, \left.\frac{\partial^2 F}{\partial t_2^2}\right|_{t=0}. \tag{4.3.4}$$

This provides all the partials, appearing in the KP equation (3.1.6).

4.3.3 Inserting partials into the integrable equation

From Theorem 3.3, the integrals $I_n(t,c)$, depending on $\beta = 2, 1, 4$, on $t = (t_1, t_2, ...)$ and on the boundary points $c = (c_1, ..., c_{2r})$ of E, relate to τ-functions, as follows:

$$\begin{aligned} I_n(t,c) &= \int_{E^n} |\Delta_n(z)|^\beta \prod_{k=1}^n \left(e^{\sum_1^\infty t_i z_k^i}\rho(z_k)dz_k\right) \\ &= \left\{\begin{array}{lll} n!\tau_n(t,c), & n \text{ arbitrary}, & \beta=2 \\ n!\tau_n(t,c), & n \text{ even}, & \beta=1 \\ n!\tau_{2n}(t/2,c), & n \text{ arbitrary}, & \beta=4 \end{array}\right. , \end{aligned} \tag{4.3.5}$$

where $\tau_n(t,c)$ satisfies the KP-like equation

$$12\frac{\tau_{n-2}(t,c)\tau_{n+2}(t,c)}{\tau_n(t,c)^2}\delta_{1,4}^\beta = (KP)_t \log \tau_n(t,c), \quad \left\{\begin{array}{l} n \text{ arbitrary for } \beta=2 \\ n \text{ even for } \beta=1,4 \end{array}\right. \tag{4.3.6}$$

with

$$(KP)_t F := \left(\left(\frac{\partial}{\partial t_1}\right)^4 + 3\left(\frac{\partial}{\partial t_2}\right)^2 - 4\frac{\partial^2}{\partial t_1 \partial t_3}\right) F + 6\left(\frac{\partial^2}{\partial t_1^2}F\right)^2.$$

Evaluating the left hand side of (4.3.6): Here $I_n(t)$ will refer to the integral (4.3.5) over the full range. For $\beta = 2$, the left hand side is zero. For $\beta = 1$, the left hand side can be evaluated in terms of the probability $P_n(E)$, as follows: taking into account $P_n := P_n(E) = I_n(0,c)/I_n(0)$,

$$\begin{aligned} 12\left.\frac{\tau_{n-2}(t,c)\tau_{n+2}(t,c)}{\tau_n(t,c)^2}\right|_{t=0} &= 12\frac{(n!)^2}{(n-2)!(n+2)!}\left.\frac{I_{n-2}(t,c)I_{n+2}(t,c)}{I_n(t,c)^2}\right|_{t=0} \\ &= 12\frac{n(n-1)}{(n+1)(n+2)}\frac{I_{n-2}(0)I_{n+2}(0)}{I_n(0)^2}\frac{P_{n-2}P_{n+2}}{P_n^2} \\ &= 12 b_n^{(1)}\frac{P_{n-2}(E)P_{n+2}(E)}{P_n^2(E)}, \end{aligned}$$

with $b_n^{(1)}$ given by[31]

$$b_n^{(1)} = \frac{n(n-1)}{(n+2)(n+1)} \frac{I_{n-2}(0)I_{n+2}(0)}{I_n(0)^2} = \begin{cases} \frac{n(n-1)}{16b^2} & \text{(Gaussian)} \\ \frac{n(n-1)(n+2a)(n+2a+1)}{16b^4} & \text{(Laguerre)} \\ \frac{Q}{Q_6^{\pm}} & \text{(Jacobi)} \end{cases} \tag{4.3.7}$$

For $\underline{\beta = 4}$, we have:

$$\begin{aligned} 12\frac{\tau_{2n-2}(t/2,c)\tau_{2n+2}(t/2,c)}{\tau_{2n}(t/2,c)^2}\bigg|_{t=0} &= 12\frac{(n!)^2}{(n-1)!(n+1)!} \frac{I_{n-1}(t,c)I_{n+1}(t,c)}{I_n(t,c)^2}\bigg|_{t=0} \\ &= 12\frac{n}{(n+1)}\frac{I_{n-1}(0)I_{n+1}(0)}{I_n(0)^2}\frac{P_{n-1}P_{n+1}}{P_n^2} \\ &= 12b_n^{(4)}\frac{P_{n-1}(E)P_{n+1}(E)}{P_n^2(E)}, \end{aligned}$$

with

$$b_n^{(4)} := \frac{(n!)^2}{(n-1)!(n+1)!}\frac{I_{n-1}(0)I_{n+1}(0)}{I_n^2(0)} = \begin{cases} \frac{2n(2n+1)}{4b^2} & \text{(Gauss)} \\ \frac{2n(2n+1)(2n+a)(2n+a-1)}{b^4} & \text{(Laguerre)} \\ \frac{Q}{Q_6^{\pm}} & \text{(Jacobi)} \end{cases} \tag{4.3.8}$$

where Q is precisely the expression appearing on the left hand side of (4.1.10), and where $Q_6^{\pm}$ is given by

$$Q_6^{\pm} = 3q\,(q+1)\,(q-3)\left(q+4\pm 4\sqrt{q+1}\right) \qquad \begin{cases} + & \text{for} \quad \beta=1 \\ - & \text{for} \quad \beta=4 \end{cases} \tag{4.3.9}$$

The exact formulae $b_n^{(4)}$ and $b_n^{(1)}$ show they satisfy the duality property (4.1.2):

$$b_n^{(4)}(a,b,n) = b_n^{(1)}(-\frac{a}{2},-\frac{b}{2},-2n).$$

[31]this calculation is based on Selberg's integrals (see Mehta [49], p 340). For instance, in the Jacobi case, one uses

$$\begin{aligned} I_n^{(\beta)} &= \int_{[-1,1]^n} \Delta_n(x)^\beta \prod_{j=1}^n (1-x_j)^a(1+x_j)^b dx_j \\ &= 2^{n(2a+2b+\beta(n-1)+2)/2}\prod_{j=0}^{n-1}\frac{\Gamma(a+j\beta/2+1)\Gamma(b+j\beta/2+1)\Gamma((j+1)\beta/2+1)}{\Gamma(\beta/2+1)\Gamma(a+b+(n+j-1)\beta/2+2)}. \end{aligned}$$

<u>*Evaluating the right hand side of (4.3.6)*</u>: From section 2.4, it also follows that $F_n(t;c) = \log I_n(t;c)$ satisfies Virasoro constraints, corresponding precisely to the situation (4.3.1), with
Gaussian ensemble[32]:

$$\begin{cases} \gamma_{1,-1} = -\frac{1}{2}, \gamma_{1,0} = \gamma_1 = 0, \delta_1 = -\frac{n}{2} \\ \gamma_{2,-1} = 0, \gamma_{2,0} = -1/2, \gamma_{2,1} = 0, \gamma_2 = -\frac{n}{4}\sigma_1, \delta_2 = 0 \\ \gamma_{3,-1} = -\frac{1}{4}\sigma_1, \gamma_{3,0} = 0, \gamma_{3,1} = -\frac{1}{2}, \gamma_{3,2} = \gamma_3 = 0, \delta_3 = -\frac{n}{4}\sigma_1. \end{cases}$$

Laguerre ensemble[33]: $\delta_1 = \delta_2 = \delta_3 = 0$, and

$$\begin{cases} \gamma_{1,-1} = 0, \gamma_{1,0} = -1, \gamma_1 = -\frac{n}{2}(\sigma_1 + a), \\ \gamma_{2,-1} = 0, \gamma_{2,0} = -\sigma_1, \gamma_{2,1} = -1, \gamma_2 = -\frac{n}{2}\sigma_1(\sigma_1 + a), \\ \gamma_{3,-1} = 0, \gamma_{3,0} = -\sigma_1\sigma_2, \gamma_{3,1} = -\sigma_2, \gamma_{3,2} = -1, \gamma_3 = -\frac{n}{2}\sigma_1\sigma_2(\sigma_1 + a). \end{cases}$$

Jacobi ensemble: see (4.3.4).

They lead to expressions for

$$\left.\frac{\partial^4 F}{\partial t_1^4}\right|_{t=0}, \left.\frac{\partial^2 F}{\partial t_2^2}\right|_{t=0}, \left.\frac{\partial^2 F}{\partial t_1 \partial t_3}\right|_{t=0}, \left.\frac{\partial^2 F}{\partial t_1^2}\right|_{t=0},$$

in terms of $\mathcal{D}_k$ and $\mathcal{B}_k$, which substituted in the right hand side of (4.3.6) - i.e. in the KP-expressions - leads to the right hand side of (4.1.4),(4.1.6), (4.1.9) and (4.1.10). In the Jacobi case, the right hand side of (4.3.6) contains the same coefficient $1/Q_6^{\pm}$ as in (4.3.9), which therefore cancels with the one appearing on the left hand side; see the expression $b_n^{1,4}$ in (4.3.7) and (4.3.8).

5 Ensembles of infinite random matrices: Fredholm determinants, as τ-functions of the KdV equation

Infinite Hermitian matrix ensembles typically relate to the Korteweg-de Vries hierarchy, itself a reduction of the KP hierarchy; a brief sketch will be necessary. The *KP-hierarchy* is given by t_n-deformations of a pseudo-differential operator[34] L: (commuting vector fields)

$$\frac{\partial L}{\partial t_n} = [(L^n)_+, L], \quad L = D + a_{-1}D^{-1} + ..., \quad \text{with} \quad D = \frac{\partial}{\partial x}. \tag{5.0.1}$$

[32] Remember from section 2.1, $\sigma_1 = \beta(n-1) + 2$.
[33] Remember from section 2.1, $\sigma_1 = \beta(n-1) + a + 2$ and $\sigma_2 = \beta(n - \frac{3}{2}) + a + 3$.
[34] In this section, given P a pseudo-differential operator, P_+ and P_- denote the differential and the (strictly) smoothing part of P respectively.

Wave and adjoint wave functions are eigenfunctions $\Psi^+(x,t;z)$ and $\Psi^-(x,t;z)$, depending on $x\in\mathbb{R},\ t\in\mathbb{C}^\infty,\ z\in\mathbb{C}$, behaving asymptotically like (5.0.3) below and satisfying:

$$\begin{aligned} z\Psi^+ = L\Psi^+, &\quad \frac{\partial\Psi^+}{\partial t_n} = (L^n)_+\Psi^+, \\ z\Psi^- = L^\top\Psi^-, &\quad \frac{\partial\Psi^-}{\partial t_n} = -(L^{\top n})_+\Psi^-. \end{aligned} \tag{5.0.2}$$

According to Sato's theory, Ψ^+ and Ψ^- have the following representation in terms of a τ-function (see [25]):

$$\begin{aligned} \Psi^\pm(x,t;z) &= e^{\pm(xz+\sum_1^\infty t_iz^i)}\frac{\tau(t\mp[z^{-1}])}{\tau(t)} \\ &= e^{\pm(xz+\sum_1^\infty t_iz^i)}(1+O(z^{-1})),\ \text{for } z\nearrow\infty, \end{aligned} \tag{5.0.3}$$

where τ satisfies and is characterized by the following bilinear relation

$$\oint e^{\sum_1^\infty (t_i-t_i')z^i}\tau(t-[z^{-1}])\tau(t'+[z^{-1}])dz = 0, \quad \text{for all } t,t'\in\mathbb{C}^\infty; \tag{5.0.4}$$

the integral is taken over a small circle around $z=\infty$. From the bilinear relation, one derives the KP-hierarchy, already mentioned in Theorem 3.1, of which the first equation reads as in (3.1.6).

We consider the p-reduced KP hierarchy, i.e., the reduction to pseudo- differential L's such that $L^p = D^p+\dots$ is a differential operator for some fixed $p\geq 2$. Then $(L^{kp})_+ = L^{kp}$ for all $k\geq 1$ and thus $\partial L/\partial t_{kp}=0$, in view of the deformation equations (5.0.1) on L. Therefore the variables $t_p, t_{2p}, t_{3p}, \dots$ are not active and can thus be set $=0$. The case $p=2$ is particularly interesting and leads to the KdV equation, upon setting all even $t_i=0$.

For the time being, take the integer $p\geq 2$ arbitrary. The arbitrary linear combinations[35]

$$\Phi^\pm(x,t;z) := \sum_{\omega\in\zeta_p} a_\omega^\pm \Psi^\pm(x,t;\omega z) \tag{5.0.5}$$

are the most general solution of the spectral problems $L^p\Phi^+ = z^p\Phi^+$ and $L^{\top p}\Phi^- = z^p\Phi^-$ respectively, leading to the definition of the kernels:

$$k_{x,t}(y,z) := \int^x dx\ \Phi^-(x,t;y)\Phi^+(x,t;z),\ \text{and } k^E_{x,t}(y,z) := k_{x,t}(y,z)I_E(z), \tag{5.0.6}$$

where the integral is taken from a fixed, but arbitrary origin in $\mathbb{R}$. In the same way that $\Psi^\pm(x,t,z)$ has a τ-function representation, so also does $k^E_{x,t}(y,z)$ have a similar representation, involving the vertex operator:

$$Y(x,t;y,z) := \sum_{\omega,\omega'\in\zeta_p} a_\omega^- a_{\omega'}^+ X(x,t;\omega y,\omega' z), \tag{5.0.7}$$

[35] $\zeta_p := \{\omega \text{ such that } \omega^p = 1\}$

where (see [25, 11, 12])

$$X(x,t;y,z) := \frac{1}{z-y} e^{(z-y)x+\sum_1^\infty (z^i-y^i)t_i} e^{\sum_1^\infty (y^{-i}-z^{-i})\frac{1}{i}\frac{\partial}{\partial t_i}}. \tag{5.0.8}$$

A condition $\sum_{\omega\in\zeta_p} \frac{a_\omega^+ a_{\bar\omega}^-}{\omega} = 0$ is needed to guarantee that the right hand side of (5.0.7) is free of singularities in the positive quadrant $\{y_i \geq 0 \text{ and } z_j \geq 0 \text{ with } i,j = 1,...,n\}$ and $\lim_{y\to z} Y(x,t;y,z)$ exists. Indeed, using Fay identities and higher degree Fay identities, one shows stepwise the following three statements, the last one being a statement about a Fredholm determinant[36]:

$$\begin{aligned} k_{x,t}(y,z) &= \frac{1}{\tau(t)} Y(x,t;y,z)\tau(t) \\ \det\Big(k_{x,t}(y_i,z_j)\Big)_{1\leq i,j\leq n} &= \frac{1}{\tau}\prod_{i=1}^k Y(x,t;y_i,z_i)\tau \\ \det(I-\lambda k^E_{x,t}) &= \frac{1}{\tau} e^{-\lambda\int_E dz\, Y(x,t;z,z)}\tau =: \frac{\tau(t,E)}{\tau(t)}. \end{aligned} \tag{5.0.9}$$

The kernel (5.0.12) at $t=0$ will define the ststistics of a random Hermitian ensemble, when the size $n \nearrow \infty$. The next theorem is precisely a statement about Fredholm determinants of kernels of the form (5.0.12); it will be identified at $t=0$ with the probability that no eigenvalue belongs to a subset E; see section 1.2. The initial condition that Virasoro annihilates τ_0, as in sections 2.1.2 (Proof of Theorem 2.1), is now replaced by the *initial condition* (5.0.11) below.

Theorem 5.1 (Adler-Shiota-van Moerbeke [11, 12]) *Consider Virasoro generators $J_\ell^{(2)}$ satisfying*

$$\frac{\partial}{\partial z} z^{\ell+1} Y(x,t;z,z) = \left[\frac{1}{2}J_\ell^{(2)}(t), Y(x,t;z,z)\right], \tag{5.0.10}$$

where $Y(x,t;z,z)$ is defined in (5.0.7), and a τ-function satisfying the Virasoro constraint, with an arbitrary constant c_{kp}:

$$\left(J_{kp}^{(2)} - c_{kp}\right)\tau = 0 \quad \text{for a fixed } k \geq -1. \tag{5.0.11}$$

Then, given the disjoint union $E \subset \mathbb{R}^+$, the Fredholm determinant of

$$K^E_{x,t}(\lambda,\lambda') := \frac{1}{p}\frac{k_{x,t}(z,z')}{z^{\frac{p-1}{2}} z'^{\frac{p-1}{2}}} I_E(\lambda'), \quad \lambda = z^p, \lambda' = z'^p, \tag{5.0.12}$$

[36] The Fredholm determinant of a kernel $A(y,z)$ is defined by

$$\det(I-\lambda A) = 1 + \sum_{m=1}^\infty (-\lambda)^m \int\limits_{z_1\leq\cdots\leq z_m} \!\!\!\cdots\!\int \det\left(A(z_i,z_j)\right)_{1\leq i,j\leq m} dz_1\ldots dz_m.$$

satisfies the following constraint for that same $k \geq -1$:

$$\left(-\sum_{i=1}^{2r} c_i^{k+1} \frac{\partial}{\partial c_i} + \frac{1}{2p} (J_{kp}^{(2)} - c_{kp}) \right) \tau \det(I - \mu K_{x,t}^E) = 0. \tag{5.0.13}$$

The generators $J_n^{(2)}$ take on the following precise form:

$$\begin{aligned} J_n^{(1)} &:= \frac{\partial}{\partial t_n} + (-n) t_{-n} \\ J_n^{(2)} &:= \sum_{i+j=n} :J_i^{(1)} J_j^{(1)}: - (n+1) J_n^{(1)} \\ &= \sum_{i+j=n} \frac{\partial^2}{\partial t_i \partial t_j} + 2 \sum_{-i+j=n} i t_i \frac{\partial}{\partial t_j} + \sum_{-i-j=n} i t_i j t_j - (n+1) J_n^{(1)}. \end{aligned} \tag{5.0.14}$$

<u>*Remark:*</u> For KdV (i.e., $p = 2$), $(L^2)^\top = L^2 = D^2 - q(x)$ holds, and thus the adjoint wave function has the simple expression: $\Psi^-(x,t;z) = \Psi^+(x,t;-z)$. In the next two examples, which deal with KdV, set

$$\Psi(x,t;z) := \Psi^+(x,t;z).$$

Example 1: Eigenvalues of large random Hermitian matrices near the "soft edge" and the Airy kernel

Remember from section 1.2, the spectrum of the Gaussian Hermitian matrix ensemble has, for large size n, its edge at $\pm\sqrt{2n}$, near which the scaling is given by $\sqrt{2} n^{1/6}$. Therefore, the eigenvalues in Theorem 5.2 must be expressed in that new scaling. Define the disjoint union $E = \cup_1^r [c_{2i-1}, c_{2i}]$, with c_{2r} possibly ∞.

Theorem 5.2 *Given the spectrum $z_1 \geq z_2 \geq \ldots$ of the large random Hermitian matrix M, define the "eigenvalues" in the new scale:*

$$u_i = 2n^{\frac{2}{3}} \left(\frac{z_i}{\sqrt{2n}} - 1 \right) \quad \textit{for } n \nearrow \infty. \tag{5.0.15}$$

The probability of the "eigenvalues"

$$P(E^c) := P(\textit{ all eigenvalues } u_i \in E^c) \tag{5.0.16}$$

satisfies the partial differential equation (setting $\mathcal{B}_k := \sum_{i=1}^{2r} c_i^{k+1} \frac{\partial}{\partial c_i}$)[37]

$$\left(\mathcal{B}_{-1}^3 - 4(\mathcal{B}_0 - \frac{1}{2}) \right) \mathcal{B}_{-1} \log P(E^c) + 6 (\mathcal{B}_{-1}^2 \log P(E^c))^2 = 0. \tag{5.0.17}$$

[37]When $c_{2r} = \infty$, that term in $\mathcal{B}_k$ is absent.

In particular, the statistics of the largest "eigenvalue" u_1 (in the new scale) is given by

$$P(u_1 \le x) = \exp\left(-\int_x^\infty (\alpha - x) g^2(\alpha) d\alpha\right), \tag{5.0.18}$$

with

$$\begin{cases} g'' = xg + 2g^3 \quad (\textbf{Painlevé II}) \\ g(x) \cong -\frac{e^{-\frac{2}{3}x^{\frac{3}{2}}}}{2\sqrt{\pi}x^{1/4}} \text{ for } x \nearrow \infty. \end{cases} \tag{5.0.19}$$

The partial differential equation (5.0.17) is due to Adler-Shiota-van Moerbeke [11, 12]. The equation (5.0.19) for the largest eigenvalue is a special case of (5.0.17), but was first derived by Tracy-Widom [64], by methods of functional analysis.

Proof: Remember from section 1.2, the statistics of the eigenvalues is governed by the Fredholm determinant of the kernel (1.2.4), for the Hermite polynomials. In the limit,

$$\lim_{n \nearrow \infty} \frac{1}{\sqrt{2}n^{1/6}} K_n\left(\sqrt{2n} + \frac{u}{\sqrt{2}n^{1/6}}, \sqrt{2n} + \frac{v}{\sqrt{2}n^{1/6}}\right) = K(u,v),$$

where

$$K(u,v) = \int_0^\infty A(x+u)A(x+v)dx, \quad A(u) = \int_{-\infty}^\infty e^{iux - x^3/3} dx. \tag{5.0.20}$$

Then

$$P(E^c) := P(\text{ all eigenvalue } u_i \in E^c) = \det(I - K(u,v)I_E(v)). \tag{5.0.21}$$

In order to compute the PDE's of this expression, with regard to the endpoints c_i of the disjoint union E, one proceeds as follows:

Consider the KdV wave function $\Psi(x,t;z)$, as in (5.0.2), with initial condition:

$$\Psi(x,t_0;z) = z^{\frac{1}{2}} A(x+z^2) = e^{xz + \frac{2}{3}z^3}(1 + O(z^{-1})) \ , \ z \to \infty, \ t_0 = (0,0,\frac{2}{3},0,...), \tag{5.0.22}$$

in terms of the Airy function[38], which, by stationary phase, has the asymptotics:

$$A(u) := \frac{1}{\sqrt{\pi}} \int_{-\infty}^\infty e^{-\frac{y^3}{3} + yu} dy = u^{-\frac{1}{4}} e^{\frac{2}{3}u^{\frac{3}{2}}}(1 + O(u^{-\frac{3}{2}})).$$

The definition of $A(u)$ is slightly changed, compared to (5.0.20). $A(u)$ satisfies the differential equation $A(y)'' = yA(y)$, and thus the wave function $\Psi(x,t_0;z)$ satisfies $(D^2 - x)\Psi(x,t_0;z) = z^2\Psi(x,t_0;z)$. Therefore $L^2|_{t=t_0} = SD^2S^{-1}|_{t=t_0} = D^2 - x$, so

[38] The i in the definition of the Airy function is omitted here.

that L^2 is a differential operator, and Ψ is a KdV wave function, with $\tau(t)$ satisfying[39] the Virasoro constraints (5.0.11) with $c_{2k} = -\frac{1}{4}\delta_{k0}$. The argument to prove these constraints is based on the fact that the linear span (a point in an infinite-dimensional Grassmannian)

$$\mathcal{W} = \operatorname{span}_{\mathbb{C}} \left\{ \psi_n(z) := e^{-\frac{2}{3}z^3} \sqrt{z}\, \frac{\partial^n}{\partial u^n} A(u) \Big|_{u=z^2},\ n = 0, 1, 2, ... \right\}$$

is invariant under multiplication by z^2 and the operator $\frac{1}{2z}(\frac{\partial}{\partial z} + 2z^2) - \frac{1}{4}z^{-2}$.

Define for $\lambda = z^2, \lambda' = z'^2$, the kernel $K_t(\lambda, \lambda')$,

$$K_t(\lambda, \lambda') := \frac{1}{2z^{\frac{1}{2}} z'^{\frac{1}{2}}} \int_0^\infty \Psi(x, t; z)\Psi(x, t; z')dx, \tag{5.0.23}$$

which flows off the Airy kernel, by (5.0.22),

$$K_{t_0}(\lambda, \lambda') = \frac{1}{2} \int_0^\infty A(x + \lambda)A(x + \lambda')dx.$$

Thus $\tau \det(I - K^E_{x,t})$ satisfies (5.0.13), with that same constant c_{kp}, for $k = -1, 0, 1, \ldots$:

$$\left(-\sum_{i=1}^{2r} c_i^{k+1} \frac{\partial}{\partial c_i} + \frac{1}{4} J^{(2)}_{pk} + \frac{1}{16}\delta_{k,0} \right) \tau \det(I - K^E_t) = 0. \tag{5.0.24}$$

Upon shifting $t_3 \mapsto t_3 + 2/3$, in view of (5.0.22), the two first Virasoro constraints for $k = -1$ and $k = 0$ read: ($\mathcal{B}_k := \sum_{i=1}^{2r} c_i^{k+1} \frac{\partial}{\partial c_i}$)

$$\begin{aligned} \mathcal{B}_{-1} \log \tau(t, E) &= \left(\frac{\partial}{\partial t_1} + \frac{1}{2} \sum_{i \geq 3} i t_i \frac{\partial}{\partial t_{i-2}} \right) \log \tau(t, E) + \frac{t_1^2}{4} \\ \mathcal{B}_0 \log \tau(t, E) &= \left(\frac{\partial}{\partial t_3} + \frac{1}{2} \sum_{i \geq 1} i t_i \frac{\partial}{\partial t_i} \right) \log \tau(t, E) + \frac{1}{16}. \end{aligned} \tag{5.0.25}$$

The same method as in section 4 enables one to express all the t-partials, appearing in the KdV equation,

$$\left(\frac{\partial^4}{\partial t_1^4} - 4\frac{\partial^2}{\partial t_1 \partial t_3} \right) \log \tau(t, E) + 6 \left(\frac{\partial^2}{\partial t_1^2} \log \tau(t, E) \right)^2 = 0,$$

[39] Although not used here, the τ-function is Kontsevich's integral:((see [44, 10])

$$\tau(t) = \frac{\int_{\mathcal{H}} dY\, e^{-\operatorname{Tr}(\frac{1}{3}Y^3 + Y^2 Z)}}{\int_{\mathcal{H}} dY\, e^{-\operatorname{Tr} Y^2 Z}} \text{ with } t_n = -\frac{1}{n} \operatorname{Tr}(Z^{-n}) + \frac{2}{3}\delta_{n,3},\ Z = \text{diagonal matrix}.$$

in terms of c-partials, which upon substitution leads to the partial differential equation $\left(\mathcal{B}_{-1}^3 - 4(\mathcal{B}_0 - \frac{1}{2})\right) f + 6(\mathcal{B}_{-1} f)^2 = 0$ (announced in (5.0.17)) for

$$f := \mathcal{B}_{-1} \log P(E^c) = \sum_1^{2r} \frac{\partial}{\partial c_i} \log P(E^c), \quad \text{where } P(E^c) = \det(I - K^E) = \frac{\tau(t,E)}{\tau(t)}.$$

When $E = (-\infty, x)$, this PDE reduces to an ODE:

$$f''' - 4xf' + 2f + 6f'^2 = 0, \quad \text{with} \quad f = \frac{d}{dx} \log P(\max_i \ \lambda_i \le x). \tag{5.0.26}$$

According to Appendix on Chazy classes (section 9) , this equation can be reduced to

$$f''^2 + 4f'(f'^2 - xf' + f) = 0, \qquad (\textbf{Painlevé II}) \tag{5.0.27}$$

which can be solved by setting

$$f' = -g^2 \text{ and } f = g'^2 - xg^2 - g^4.$$

An easy computation shows g satisfies the equation $g'' = 2g^3 + xg$ (Painlevé II), thus leading to (5.0.19).

Example 2: Eigenvalues of large random Laguerre Hermitian matrices near the "hard edge" and the Bessel kernel

Consider the ensemble of $n \times n$ random matrices for the Laguerre probability distribution, thus corresponding to (1.1.9) with $\rho(dz) = z^{\nu/2} e^{-z/2} dz$. Remember from section 1.2, the density of eigenvalues near the "hard edge" $z = 0$ is given by $4n$ for very large n. At this edge, the kernel (1.2.4) with Laguerre polynomials p_n tends to the Bessel kernel [52, 30]:

$$\lim_{n \nearrow \infty} \frac{1}{4n} K_n^{(\nu)} \left(\frac{u}{4n}, \frac{v}{4n} \right) = K^{(\nu)}(u,v) := \frac{1}{2} \int_0^1 x J_\nu(xu) J_\nu(xv) dx. \tag{5.0.28}$$

Therefore, the eigenvalues in the theorem below will be expressed in that new scaling. Define, as before, the disjoint union $E = \cup_1^r [c_{2i-1}, c_{2i}]$.

Theorem 5.3 *Given the spectrum $0 \le z_1 \le z_2 \le \dots$ of the large random Laguerre-distributed Hermitian matrix M, define the "eigenvalues" in the new scale:*

$$u_i = 4nz_i \quad \textit{for } n \nearrow \infty. \tag{5.0.29}$$

The statistics of the "eigenvalues"

$$P(E^c) := P(\text{ all "eigenvalues" } u_i \in E^c) \tag{5.0.30}$$

leads to the following PDE for $F = \log P(E^c)$: (setting $\mathcal{B}_k := \sum_{i=1}^{2r} c_i^{k+1} \frac{\partial}{\partial c_i}$)

$$\left(\mathcal{B}_0^4 - 2\mathcal{B}_0^3 + (1-\nu^2)\mathcal{B}_0^2 + \mathcal{B}_1\left(\mathcal{B}_0 - \frac{1}{2}\right)\right) F - 4(\mathcal{B}_0 F)(\mathcal{B}_0^2 F) + 6(\mathcal{B}_0^2 F)^2 = 0. \quad (5.0.31)$$

In particular, for very large n, the statistics of the smallest eigenvalue is governed by

$$P(u_1 \geq x) = \exp\left(-\int_0^x \frac{f(u)}{u} du\right), \qquad u_1 \sim 4nz_1,$$

with f satisfying

$$(xf'')^2 - 4\big(xf' - f\big)f'^2 + \big((x-\nu^2)f' - f\big)f' = 0. \ \textbf{(Painlevé V)} \qquad (5.0.32)$$

The equation (5.0.32) for the smallest eigenvalue, first derived by Tracy-Widom [64], by methods of functional analysis, is a special case of the partial differential equation (5.0.31), due to [11, 12].

<u>*Remark:*</u> This same theorem would hold for the Jacobi ensemble, near the "hard edges" $z = \pm 1$.

Proof: Define a wave function $\Psi(x, t; z)$, flowing off

$$\Psi(x, 0; z) = e^{xz} B(-xz) = e^{xz}(1 + O(z^{-1}),$$

where $B(z)$ is the Bessel function[40]

$$B(z) = \varepsilon\sqrt{z} e^z H_\nu(iz) = \frac{e^z 2^{\nu+1/2}}{\Gamma(-\nu+1/2)} \int_1^\infty \frac{z^{-\nu+1/2} e^{-uz}}{(u^2-1)^{\nu+1/2}} du = 1 + O(z^{-1}).$$

As the operator

$$L^2|_{t=0} = D^2 - \frac{\nu^2 - 1/4}{x^2}$$

is a differential operator, we are in the KdV situation; again one may assume $t_2 = t_4 = \cdots = 0$ and we have

$$\Psi^-(x, t; -z) = \Psi^+(x, t; z) = e^{xz + \sum t_i z^i} \frac{\tau(t - [z^{-1}])}{\tau(t)},$$

in terms of a τ-function[41] satisfying the following Virasoro constraints

$$J_{2k}^{(2)} \tau = ((2\nu)^2 - 1)\delta_{k0}\tau. \qquad (5.0.33)$$

[40] $\varepsilon = i\sqrt{\pi/2}\, e^{i\pi\nu/2}$, $-1/2 < \nu < 1/2$.

[41] $\tau(t)$ is given by the Adler-Morozov-Shiota-van Moerbeke double Laplace matrix transform, with t_n given in a similar way as in footnote 32 (see [10]):

$$\tau(t) = c(t) \int_{\mathcal{H}_N^+} dX \det X^{\nu-1/2} e^{-\operatorname{Tr}(Z^2 X)} \int_{\mathcal{H}_N^+} dY S_0(Y) e^{-\operatorname{Tr}(XY^2)}.$$

Set $p=2$, $a_1^- = a_{-1}^+ = \frac{1}{4\pi} i e^{i\pi\nu/2}$ and $a_{-1}^- = a_1^+ = \frac{1}{4\pi} e^{-i\pi\nu/2}$ in (5.0.5); this defines the kernel (5.0.6) and so (5.0.12), which in terms of $\lambda = z^2$ and $\lambda' = z'^2$, takes on the form:

$$
\begin{aligned}
K_{x,t}^{(\nu)}(\lambda,\lambda') &= \frac{1}{4\pi\sqrt{zz'}} \int^x \left(ie^{\frac{i\pi\nu}{2}} \Psi^*(x,t,z) + e^{-\frac{i\pi\nu}{2}} \Psi^*(x,t,-z) \right) \cdot \\
&\qquad \cdot \left(e^{-\frac{i\pi\nu}{2}} \Psi(x,t,z') + ie^{\frac{i\pi\nu}{2}} \Psi(x,t,-z') \right) dx,
\end{aligned}
$$

which flows off the **Bessel kernel**,

$$
\begin{aligned}
K_{x,0}^{(\nu)}(\lambda,\lambda') &= \frac{1}{2}\int_0^x x J_\nu(x\sqrt{\lambda}) J_\nu(x\sqrt{\lambda'}) dx. \\
&= \frac{J_\nu(\sqrt{\lambda})\sqrt{\lambda'} J_\nu'(\sqrt{\lambda'}) - J_\nu(\sqrt{\lambda'})\sqrt{\lambda} J_\nu'(\sqrt{\lambda})}{2(\lambda-\lambda')} \quad \text{for} \quad x=1.
\end{aligned}
$$

The Fredholm determinant satisfies for $E \subset \mathbb{R}_+$ and for $k=0,1,\dots$:

$$
\left(-\sum_1^{2r} c_i^{k+1} \frac{\partial}{\partial c_i} + \frac{1}{4} J_{2k}^{(2)} + \left(\frac{1}{4} - \nu^2\right) \delta_{k,0} \right) \left(\tau \det(I - K_{x,t}^{(\nu)E} \right) = 0. \tag{5.0.34}
$$

Upon shifting $t_1 \mapsto t_1 + \sqrt{-1}$ and using the same $\mathcal{B}_i$ as in (5.0.25), the equations for $k=0$ and $k=1$ read

$$
\begin{aligned}
\mathcal{B}_0 \log \tau(t,E) &= \frac{1}{2}\left(\sum_{i\geq 1} it_i \frac{\partial}{\partial t_i} + \sqrt{-1}\frac{\partial}{\partial t_1} \right) \log \tau(t,E) + \frac{1}{4}(\frac{1}{4} - \nu^2) \\
\mathcal{B}_1 \log \tau(t,E) &= \frac{1}{2}\left(\sum_{i\geq 1} it_i \frac{\partial}{\partial t_{i+2}} + \frac{1}{2}\frac{\partial^2}{\partial t_1^2} + \sqrt{-1}\frac{\partial}{\partial t_3} + \frac{1}{2}\frac{\partial}{\partial t_1} \right) \log \tau(t,E).
\end{aligned} \tag{5.0.35}
$$

Expressing the t-partials (5.0.20), appearing in the KdV-equation at $t=0$ (see formula below (5.0.25)) in terms of the c-partials applied to $\log \tau(0,E)$, leads to the following PDE for $F = \log P(E^c)$:

$$
\left(\mathcal{B}_0^4 - 2\mathcal{B}_0^3 + (1-\nu^2)\mathcal{B}_0^2 + \mathcal{B}_1\left(\mathcal{B}_0 - \frac{1}{2}\right) \right) F - 4(\mathcal{B}_0 F)(\mathcal{B}_0^2 F) + 6(\mathcal{B}_0^2 F)^2 = 0. \tag{5.0.36}
$$

Specializing this equation to the interval $E=(0,x)$ leads to an ODE for $f := -x\,\partial F/\partial x$, namely

$$
f''' + \frac{1}{x} f'' - \frac{6}{x} f'^2 + \frac{4}{x^2} f f' + \frac{(x-\nu^2)}{x^2} f' - \frac{1}{2x^2} f = 0, \tag{5.0.37}
$$

which is an equation of the type (9.0.1); changing $x \curvearrowright -x$ and $f \curvearrowright -f$ leads again to an equation of type (9.0.1), with $P(x) = x$, $4Q(x) = -x - \nu^2$ and $R=0$. According to

Cosgrove-Scoufis [24] (see the Appendix on Chazy classes), this equation can be reduced to the equation (9.0.2), with the same P, Q, R and with $c = 0$. Since $P(x) = x$, this equation is already in one of the canonical forms (9.0.3), which upon changing back x and f, leads to

$$(xf'')^2 + 4\big(-xf' + f\big)f'^2 + \big((x-\nu^2)f' - f\big)f' = 0. \quad \textbf{(Painlevé V)}$$

Example 3: Eigenvalues of large random Gaussian Hermitian matrices in the bulk and the sine kernel

Setting $\nu = \pm 1/2$ yields kernels related to the sine kernel:

$$\begin{aligned}
K_{x,0}^{(+1/2)}(y^2, z^2) &= \frac{1}{\pi}\int_0^x \frac{\sin xy \ \sin xz}{y^{1/2}z^{1/2}}dx \\
&= \frac{1}{2\pi}\left(\frac{\sin x(y-z)}{y-z} - \frac{\sin x(y+z)}{y+z}\right) \\
K_{x,0}^{(-1/2)}(y^2, z^2) &= \frac{1}{\pi}\int_0^x \frac{\cos xy \ \cos xz}{y^{1/2}z^{1/2}}dx \\
&= \frac{1}{2\pi}\left(\frac{\sin x(y-z)}{y-z} + \frac{\sin x(y+z)}{y+z}\right).
\end{aligned}$$

Therefore the sine-kernel obtained in the context of the bulk-scaling limit (see 1.2.7) is the sum $K_{x,0}^{(+1/2)} + K_{x,0}^{(-1/2)}$. Expressing the Fredholm determinant of this sum in terms of the Fredholm determinants of each of the parts, leads to the Painlevé V equation (1.2.8).

6 Coupled random Hermitian ensembles

Consider a product ensemble $(M_1, M_2) \in \mathcal{H}_n^2 := \mathcal{H}_n \times \mathcal{H}_n$ of $n \times n$ Hermitean matrices, equipped with a Gaussian probability measure,

$$c_n dM_1 dM_2\, e^{-\frac{1}{2}\mathrm{Tr}(M_1^2 + M_2^2 - 2cM_1M_2)}, \tag{6.0.1}$$

where $dM_1 dM_2$ is Haar measure on the product $\mathcal{H}_n^2$, with each dM_i,

$$dM_1 = \Delta_n^2(x)\prod_1^n dx_i dU \text{ and } dM_2 = \Delta_n^2(y)\prod_1^n dy_i dU \tag{6.0.2}$$

decomposed into radial and angular parts. In terms of the coupling constant c, appearing in (6.0.1), and the boundary of the set

$$E = E_1 \times E_2 := \cup_{i=1}^r [a_{2i-1}, a_{2i}] \times \cup_{i=1}^s [b_{2i-1}, b_{2i}] \subset \mathbb{R}^2, \tag{6.0.3}$$

define differential operators $\mathcal{A}_k$, $\mathcal{B}_k$ of "weight" k,

$$\mathcal{A}_1=\frac{1}{c^2-1}\left(\sum_1^r\frac{\partial}{\partial a_j}+c\sum_1^s\frac{\partial}{\partial b_j}\right)\quad \mathcal{B}_1=\frac{1}{1-c^2}\left(c\sum_1^r\frac{\partial}{\partial a_j}+\sum_1^s\frac{\partial}{\partial b_j}\right)$$
$$\mathcal{A}_2=\sum_{j=1}^r a_j\frac{\partial}{\partial a_j}-c\frac{\partial}{\partial c}\qquad \mathcal{B}_2=\sum_{j=1}^s b_j\frac{\partial}{\partial b_j}-c\frac{\partial}{\partial c},$$

forming a closed Lie algebra[42]. The following theorem follows, via similar methods, from the Virasoro constraints (3.3.5) and the 2-Toda equation (3.3.6):

Theorem 6.1 (**Gaussian probability**) (Adler-van Moerbeke [3]) *The joint statistics*

$$\begin{aligned}P_n(M\in\mathcal{H}_n^2(E_1\times E_2)) &= \frac{\displaystyle\iint_{\mathcal{H}_n^2(E_1\times E_2)} dM_1dM_2\, e^{-\frac12\operatorname{Tr}(M_1^2+M_2^2-2cM_1M_2)}}{\displaystyle\iint_{\mathcal{H}_n^2} dM_1dM_2\, e^{-\frac12\operatorname{Tr}(M_1^2+M_2^2-2cM_1M_2)}}\\ &= \frac{\displaystyle\iint_{E^n}\Delta_n(x)\Delta_n(y)\prod_{k=1}^n e^{-\frac12(x_k^2+y_k^2-2cx_ky_k)}dx_kdy_k}{\displaystyle\iint_{\mathbb{R}^{2n}}\Delta_n(x)\Delta_n(y)\prod_{k=1}^n e^{-\frac12(x_k^2+y_k^2-2cx_ky_k)}dx_kdy_k}\end{aligned}$$

satisfies the non-linear third-order partial differential equation[43] *(independent of n) ($F_n:=\frac1n\log P_n(E)$):*

$$\left\{\mathcal{B}_2\mathcal{A}_1F_n\ ,\ \mathcal{B}_1\mathcal{A}_1F_n+\frac{c}{c^2-1}\right\}_{\mathcal{A}_1}-\left\{\mathcal{A}_2\mathcal{B}_1F_n\ ,\ \mathcal{A}_1\mathcal{B}_1F_n+\frac{c}{c^2-1}\right\}_{\mathcal{B}_1}=0. \quad (6.0.4)$$

7 Random permutations

The purpose of this section is to show that the generating function of the probability

$$P(L(\pi_n)\le\ell)=\frac{1}{n!}\#\{\pi_n\in S_n\mid L(\pi_n)\le\ell\}$$

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$$[\mathcal{A}_1,\mathcal{B}_1]=0\quad[\mathcal{A}_1,\mathcal{A}_2]=\frac{1+c^2}{1-c^2}\mathcal{A}_1\quad[\mathcal{A}_2,\mathcal{B}_1]=\frac{2c}{1-c^2}\mathcal{A}_1$$
$$[\mathcal{A}_2,\mathcal{B}_2]=0\quad[\mathcal{A}_1,\mathcal{B}_2]=\frac{-2c}{1-c^2}\mathcal{B}_1\quad[\mathcal{B}_1,\mathcal{B}_2]=\frac{1+c^2}{1-c^2}\mathcal{B}_1.$$

43 in terms of the Wronskian $\{f,g\}_X=Xf.g-f.Xg$, with regard to a first order differential operator X.

is closely related to a special solution of the Painlevé V equation, with peculiar initial condition. Remember from section 1.4, $L(\pi_n)$ is the length of the longest increasing sequence in the permutation π_n.

Theorem 7.1 *For every $\ell \geq 0$,*

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} P(L(\pi_n) \leq \ell) = \int_{U(\ell)} e^{\sqrt{x}\ tr(M+\bar{M})} dM = \exp \int_0^x \log\left(\frac{x}{u}\right) g_\ell(u) du, \tag{7.0.1}$$

with g_ℓ satisfying the initial value problem for **Painlevé V***:*

$$\begin{cases} g'' - \dfrac{g'^2}{2}\left(\dfrac{1}{g-1} + \dfrac{1}{g}\right) + \dfrac{g'}{u} + \dfrac{2}{u} g(g-1) - \dfrac{\ell^2}{2u^2}\dfrac{g-1}{g} = 0 \\ \\ g_\ell(u) = 1 - \dfrac{u^\ell}{(\ell!)^2} + O(u^{\ell+1}), \quad near\ u = 0. \end{cases} \tag{7.0.2}$$

The Painlevé V equation for this integral was first found by Tracy-Widom [68]. This systematic derivation and the initial condition are due to Adler-van Moerbeke [7].

Proof: Upon inserting $(t_1, t_2, ...)$ and $(s_1, s_2, ...)$ variables in the $U(n)$-integral (7.0.1), the integral

$$\begin{aligned} I_n(t,s) &= \int_{U(n)} e^{Tr \sum_1^\infty (t_i M^i - s_i \bar{M}^i)} dM \qquad (7.0.3) \\ &= n! \det\left(\int_{S^1} z^{k-\ell} e^{\sum_1^\infty (t_i z^i - s_i z^{-i})} \frac{dz}{2\pi i z}\right)_{0 \leq k,\ell \leq n-1} = n! \tau_n(t,s) \end{aligned}$$

puts us in the conditions of Theorem 3.5. It deals with semi-infinite matrices L_1 and $hL_2^\top h^{-1}$ of "rank 2", having diagonal elements

$$b_n := \frac{\partial}{\partial t_1} \log \frac{\tau_n}{\tau_{n-1}} = (L_1)_{n-1,n-1} \quad \text{and} \quad b_n^* := -\frac{\partial}{\partial s_1} \log \frac{\tau_n}{\tau_{n-1}} = (hL_2^\top h^{-1})_{n-1,n-1}.$$

To summarize Theorem 3.5, $I_n(t,s)$ satisfies the following three types of identities:

(i) **Virasoro**: ($F := \log\tau_n$) (see (3.4.7))

$$\begin{aligned}
0 = \frac{\mathcal{V}_{-1}\tau_n}{\tau_n} &= \left(\sum_{i\geq 1}(i+1)t_{i+1}\frac{\partial}{\partial t_i} - \sum_{i\geq 2}(i-1)s_{i-1}\frac{\partial}{\partial s_i} + n\frac{\partial}{\partial s_1}\right)F + nt_1 \\
0 = \frac{\mathcal{V}_0\tau_n}{\tau_n} &= \sum_{i\geq 1}\left(it_i\frac{\partial}{\partial t_i} - is_i\frac{\partial}{\partial s_i}\right)F \\
0 = \frac{\mathcal{V}_1\tau_n}{\tau_n} &= \left(-\sum_{i\geq 1}(i+1)s_{i+1}\frac{\partial}{\partial s_i} + \sum_{i\geq 2}(i-1)t_{i-1}\frac{\partial}{\partial t_i}\frac{\partial}{\partial t_1}\right)F + ns_1. \\
0 = \frac{\partial}{\partial t_1}\frac{\mathcal{V}_{-1}\tau_n}{\tau_n} &= \left(\sum_{i\geq 1}(i+1)t_{i+1}\frac{\partial^2}{\partial t_1\partial t_i} - \sum_{i\geq 2}(i-1)s_{i-1}\frac{\partial^2}{\partial t_1\partial s_i} + n\frac{\partial^2}{\partial t_1\partial s_1}\right)F + n
\end{aligned} \tag{7.0.4}$$

(ii) **two-Toda**: (see (3.4.8))

$$\begin{aligned}
\frac{\partial^2\log\tau_n}{\partial s_2\partial t_1} &= -2\frac{\partial}{\partial s_1}\log\frac{\tau_n}{\tau_{n-1}}\frac{\partial^2}{\partial s_1\partial t_1}\log\tau_n - \frac{\partial^3}{\partial s_1^2\partial t_1}\log\tau_n \\
&= 2b_n^*\frac{\partial^2}{\partial s_1\partial t_1}\log\tau_n - \frac{\partial^3}{\partial s_1^2\partial t_1}\log\tau_n,
\end{aligned} \tag{7.0.5}$$

(iii) **Toeplitz**: (see (3.4.11))

$$\begin{aligned}
\mathcal{T}(\tau)_n &= \frac{\partial}{\partial t_1}\log\frac{\tau_n}{\tau_{n-1}}\frac{\partial}{\partial s_1}\log\frac{\tau_n}{\tau_{n-1}} \\
&\quad + \left(1 + \frac{\partial^2}{\partial s_1\partial t_1}\log\tau_n\right)\left(1 + \frac{\partial^2}{\partial s_1\partial t_1}\log\tau_n - \frac{\partial}{\partial s_1}\left(\frac{\partial}{\partial t_1}\log\frac{\tau_n}{\tau_{n-1}}\right)\right) \\
&= -b_nb_n^* + \left(1 + \frac{\partial^2}{\partial s_1\partial t_1}\log\tau_n\right)\left(1 + \frac{\partial^2}{\partial s_1\partial t_1}\log\tau_n - \frac{\partial}{\partial s_1}b_n\right) = 0.
\end{aligned} \tag{7.0.6}$$

Defining the locus $\mathcal{L} = \{$ all $t_i = s_i = 0$, except $t_1, s_1 \neq 0\}$, and using the second relation (7.0.4), we have on $\mathcal{L}$,

$$\frac{\mathcal{V}_0\tau_n}{\tau_n}\Big|_{\mathcal{L}} = \left(t_1\frac{\partial}{\partial t_1} - s_1\frac{\partial}{\partial s_1}\right)\log\tau_n\Big|_{\mathcal{L}} = 0,$$

implying $\tau_n(t,s)\Big|_{\mathcal{L}}$ is a function of $x := -t_1s_1$ only. Therefore we may write $\tau_n\Big|_{\mathcal{L}} = \tau_n(x)$, and so, along $\mathcal{L}$, we have $\frac{\partial}{\partial t_1} = -s_1\frac{\partial}{\partial x}$, $\quad\frac{\partial}{\partial s_1} = -t_1\frac{\partial}{\partial x}$, $\quad\frac{\partial^2}{\partial t_1\partial s_1} = -\frac{\partial}{\partial x}x\frac{\partial}{\partial x}$. Setting

$$f_n(x) = \frac{\partial}{\partial x}x\frac{\partial}{\partial x}\log\tau_n(x) = -\frac{\partial^2}{\partial t_1\partial s_1}\log\tau_n(t,s)\Big|_{\mathcal{L}}, \tag{7.0.7}$$

and using $x = -t_1 s_1$, the two-Toda relation (7.0.5) takes on the form

$$\begin{aligned} s_1 \frac{\partial^2 \log \tau_n}{\partial s_2 \partial t_1}\Big|_{\mathcal{L}} &= s_1 \left(2b_n^* \frac{\partial^2}{\partial s_1 \partial t_1} \log \tau_n - \frac{\partial}{\partial s_1} \left(\frac{\partial^2 \log \tau_n}{\partial s_1 \partial t_1} \right) \right) \\ &= x \left(2 \frac{b_n^*}{t_1} f_n + f_n' \right). \end{aligned}$$

Setting relation (7.0.7) into the Virasoro relations (7.0.4) yields

$$\begin{aligned} 0 = \frac{\mathcal{V}_0 \tau_n}{\tau_n} - \frac{\mathcal{V}_0 \tau_{n-1}}{\tau_{n-1}}\Big|_{\mathcal{L}} &= \left(t_1 \frac{\partial}{\partial t_1} - s_1 \frac{\partial}{\partial s_1} \right) \log \frac{\tau_n}{\tau_{n-1}}\Big|_{\mathcal{L}} = t_1 b_n + s_1 b_n^* \\ 0 = \frac{\partial}{\partial t_1} \frac{\mathcal{V}_{-1} \tau_n}{\tau_n}\Big|_{\mathcal{L}} &= \left(-s_1 \frac{\partial^2}{\partial s_2 \partial t_1} + n \frac{\partial^2}{\partial t_1 \partial s_1} \right) \log \tau_n \Big|_{\mathcal{L}} + n \\ &= -x \left(2 \frac{b_n^*}{t_1} f_n(x) + f_n'(x) \right) + n(-f_n(x) + 1). \end{aligned}$$

This is a system of two linear relations in b_n and b_n^*, whose solution, together with its derivatives, are given by:

$$\frac{b_n^*}{t_1} = -\frac{b_n}{s_1} = -\frac{n(f_n - 1) + x f_n'}{2x f_n}, \quad \frac{\partial b_n}{\partial s_1} = \frac{\partial}{\partial x} x \frac{b_n}{s_1} = \frac{x(f_n f_n'' - f_n'^2) + (f_n + n) f_n'}{2 f_n^2}.$$

Substituting the result into the Toeplitz relation (7.0.6), namely

$$b_n b_n^* = (1 - f_n) \left(1 - f_n - \frac{\partial}{\partial s_1} b_n \right),$$

leads to f_n satisfying Painlevé equation (7.0.2), with $g = f_n$, as in (7.0.7) and $u = x$. Note, along the locus $\mathcal{L}$, we may set $t_1 = \sqrt{x}$ and $s_1 = -\sqrt{x}$, since it respects $t, s_1 = -x$. Thus, $I_n(t, s)|_{\mathcal{L}}$ equals (7.0.1).

The initial condition (7.0.2) follows from the fact that as long as $0 \leq n \leq \ell$, the inequality $L(\pi_n) \leq \ell$ is always verified, and so

$$\begin{aligned} \sum_0^\infty \frac{x^n}{(n!)^2} \#\{\pi \in S_n \mid L(\pi_n) \leq \ell\} &= \sum_0^\ell \frac{x^n}{n!} + \frac{x^{\ell+1}}{(\ell+1)!^2}((\ell+1)! - 1) + O(x^{\ell+2}) \\ &= \exp \left(x - \frac{x^{\ell+1}}{(\ell+1)!^2} + O(x^{\ell+2}) \right), \end{aligned}$$

thus proving Proposition 7.1. ∎

<u>*Remark:*</u> Setting

$$f_n(x) = \frac{g(x)}{g(x) - 1}$$

leads to standard Painlevé V, with $\alpha = \delta = 0$, $\beta = -n^2/2, \gamma = -2$.

8 Random involutions

This section deals with a generating function for the distribution of the length of the longest increasing sequence of a fixed-point free random involution π^0_{2k}, with the uniform distribution:

$$P\left(L(\pi^0_{2k}) \leq \ell+1,\ \pi^0_{2k} \in S^0_{2k}\right) = \frac{2^k k!}{(2k)!}\#\{\pi^0_{2k} \in S^0_{2k} \bigm| L(\pi^0_{2k}) \leq \ell+1\}.$$

Proposition 8.1 (Adler-van Moerbeke [7]) *The generating function*

$$\begin{aligned} & 2\sum_{k=0}^{\infty} \frac{(x^2/2)^k}{k!} P(L(\pi^0_{2k}) \leq \ell+1) \\ & = E_{O(\ell+1)_-} e^{xTrM} + E_{O(\ell+1)_+} e^{xTrM} \\ & = \exp\left(\int_0^x \frac{f_\ell^-(u)}{u} du\right) + \exp\left(\int_0^x \frac{f_\ell^+(u)}{u} du\right), \end{aligned} \tag{8.0.1}$$

where $f = f_\ell^\pm$, satisfies the initial value problem:

$$\left\{ \begin{array}{l} f''' + \dfrac{1}{u} f'' + \dfrac{6}{u} f'^2 - \dfrac{4}{u^2} f f' - \dfrac{16u^2+\ell^2}{u^2} f' + \dfrac{16}{u} f + \dfrac{2(\ell^2-1)}{u} = 0 \\ \\ f_\ell^\pm(u) = u^2 \pm \dfrac{u^{\ell+1}}{\ell!} + O(u^{\ell+2}),\ \textit{near } u=0. \qquad (\textbf{Painlevé V}) \end{array} \right. \tag{8.0.2}$$

Proof: The first equality in (8.0.1) follows immediately from proposition 1.1. The results of section 1.3 lead to

$$\int_{O(2n+1)_\pm} e^{xTrM} dM = e^{\pm x} \int_{[-1,1]^n} \Delta_n(z)^2 \prod_{k=1}^n e^{2xz_k}(1-z_k)^a(1+z_k)^b dz_k, \tag{8.0.3}$$

with $a = \pm 1/2, b = \mp 1/2$, (with corresponding signs). Inserting t_i's in the integral, the perturbed integral, with $e^{\pm x}$ removed and with $t_1 = 2x$, reads

$$I_n(t) = \int_{[-1,1]^n} \Delta_n(z)^2 \prod_{k=1}^n (1-z_k)^a(1+z_k)^b e^{\sum_1^\infty t_i z_k^i} dz_k = n!\tau_n(t); \tag{8.0.4}$$

this is precisely integral (3.1.4) of section 3.1.1 and thus it satisfies the Virasoro constraints (3.1.5), but without boundary contribution $\mathcal{B}_i F$. Explicit Virasoro expressions appear in (2.1.35), upon setting $\beta = 2$. Also, $\tau_n(t)$, as in (3.1.4), (see Theorem 3.1)

satisfies the KP equation (3.1.6). Differentiating the Virasoro constraints in t_1 and t_2, and restricting to the locus

$$\mathcal{L} := \{t_1 = x, \text{ all other } t_i = 0\},$$

lead to a linear system of five equations, with $b_0 = a - b,\ b_1 = a + b$,

$$\begin{aligned}
&\frac{1}{I_n}\left(\mathbb{J}^{(2)}_{k+2} - \mathbb{J}^{(2)}_{k} + b_0\mathbb{J}^{(1)}_{k+1} + b_1\mathbb{J}^{(1)}_{k+2}\right) I_n\Big|_{\mathcal{L}} = 0, \quad k = -1, 0\\
&\frac{\partial}{\partial t_1}\frac{1}{I_n}\left(\mathbb{J}^{(2)}_{k+2} - \mathbb{J}^{(2)}_{k} + b_0\mathbb{J}^{(1)}_{k+1} + b_1\mathbb{J}^{(1)}_{k+2}\right) I_n\Big|_{\mathcal{L}} = 0, \quad k = -1, 0\\
&\frac{\partial}{\partial t_2}\frac{1}{I_n}\left(\mathbb{J}^{(2)}_{k+2} - \mathbb{J}^{(2)}_{k} + b_0\mathbb{J}^{(1)}_{k+1} + b_1\mathbb{J}^{(1)}_{k+2}\right) I_n\Big|_{\mathcal{L}} = 0, \quad k = -1
\end{aligned}$$

in five unknowns ($F_n = \log \tau_n$)

$$\frac{\partial F_n}{\partial t_2}\Big|_{\mathcal{L}}, \quad \frac{\partial F_n}{\partial t_3}\Big|_{\mathcal{L}}, \quad \frac{\partial^2 F_n}{\partial t_1 \partial t_2}\Big|_{\mathcal{L}}, \quad \frac{\partial^2 F_n}{\partial t_1 \partial t_3}\Big|_{\mathcal{L}}, \quad \frac{\partial^2 F_n}{\partial t_2^2}\Big|_{\mathcal{L}}.$$

Setting $t_1 = x$ and $F_n' = \partial F_n/\partial x$, the solution is given by the following expressions,

$$\begin{aligned}
\frac{\partial F_n}{\partial t_2}\Big|_{\mathcal{L}} &= -\frac{1}{x}\Big((2n+b_1)F_n' + n(b_0 - x)\Big)\\
\frac{\partial F_n}{\partial t_3}\Big|_{\mathcal{L}} &= -\frac{1}{x^2}\Big(x\left(F_n'' + F_n'^2 + (b_0 - x)F_n' + n(n+b_1)\right) - (2n+b_1)\left((2n+b_1)F_n' + b_0 n\right)\Big)\\
\frac{\partial^2 F_n}{\partial t_1 \partial t_2}\Big|_{\mathcal{L}} &= -\frac{1}{x^2}\Big((2n+b_1)(xF_n'' - F_n') - b_0 n\Big)\\
\frac{\partial^2 F_n}{\partial t_1 \partial t_3}\Big|_{\mathcal{L}} &= -\frac{1}{x^3}\Big(x^2(F_n''' + 2F_n'F_n'') - x\left((x^2 - b_0 x + 1)F_n'' + F_n'^2 + b_0 F_n' + (2n+b_1)^2 F_n''\right.\\
&\qquad\left. + n(n+b_1)\right) + 2(2n+b_1)^2 F_n' + 2b_0 n(2n+b_1)\Big)\\
\frac{\partial^2 F_n}{\partial t_2^2}\Big|_{\mathcal{L}} &= \frac{1}{x^3}\Big(x\left(2F_n'^2 + 2b_0 F_n' + ((2n+b_1)^2 + 2)F_n'' + 2n(n+b_1)\right)\\
&\qquad - 3(2n+b_1)^2 F_n' - 3b_0 n(2n+b_1)\Big).
\end{aligned}$$

Putting these expressions into KP and setting $t_1 = x$, one finds:

$$\begin{aligned}
0 &= \left(\left(\frac{\partial}{\partial t_1}\right)^4 + 3\left(\frac{\partial}{\partial t_2}\right)^2 - 4\frac{\partial^2}{\partial t_1 \partial t_3}\right) F_n + 6\left(\frac{\partial^2}{\partial t_1^2} F_n\right)^2\\
&= \frac{1}{x^3}\Big(\ x^3 F'''' + 4x^2 F''' + x\left(-4x^2 + 4b_0 x + 2 - (2n+b_1)^2\right) F'' + 8x^2 F' F''\\
&\quad + 6x^3 F''^2 + 2xF'^2 + \left(2b_0 x - (2n+b_1)^2\right) F' + n(2x - b_0)(n+b_1) - b_0 n^2\Big).
\end{aligned}$$

Finally, the function $H(x) := x\frac{d}{dx}F(x) = x\frac{d}{dx}\log\tau_n(x)$ satisfies

$$x^2H''' + xH'' + 6xH'^2 - \left(4H + 4x^2 - 4bx + (2n+a)^2\right)H' + (4x-2b)H \\ +2n(n+a)x - bn(2n+a) = 0. \tag{8.0.5}$$

This 3rd order equation is Cosgrove's [24, 23] equation, with $P = x$, $4Q = -4x^2 + 4bx - (2n+a)^2$, $2R = 2n(n+a)x - bn(2n+a)$. So, this 3rd order equation can be transformed into the Painlevé V equation (9.0.3) in the appendix. The boundary condition $f(0) = 0$ follows from the definition of H above, whereas, after an elementary, but tedious computation, $f'(0) = f''(0) = 0$ follows from the differential equation (8.0.5) and the Aomoto extension [15] (see Mehta [49], p. 340) of Selberg's integral:[44]

$$\frac{\int_0^1 \cdots \int_0^1 x_1 \ldots x_m \, |\Delta(x)|^\beta \prod_{j=1}^n x_j^\gamma (1-x_j)^\delta dx_1 ... dx_n}{\int_0^1 \cdots \int_0^1 |\Delta(x)|^\beta \prod_{j=1}^n x_j^\gamma (1-x_j)^\delta dx_1 ... dx_n} = \prod_{j=1}^m \frac{\gamma + 1 + (n-j)\beta/2}{\gamma + \delta + 2 + (2n-j-1)\beta/2}.$$

However, the initial condition (8.0.2) is a much stronger statement, again stemming from the fact that as long as $0 \le n \le \ell$, the inequality $L(\pi_n) \le \ell$ is trivially verified, thus leading to

$$E_{O_\pm(\ell+1)} e^{xTrM} = \exp\left(\frac{x^2}{2} \pm \frac{x^{\ell+1}}{(\ell+1)!} + O(x^{\ell+2})\right),$$

ending the proof of Proposition 8.1. ∎

9 Appendix: Chazy classes

Most of the differential equations encountered in this survey belong to the general Chazy class

$$f''' = F(z, f, f', f''), \text{ where } F \text{ is rational in } f, f', f'' \text{ and locally analytic in z,}$$

subjected to the requirement that the general solution be free of movable branch points; the latter is a branch point whose location depends on the integration constants. In his classification Chazy found thirteen cases, the first of which is given by

$$f''' + \frac{P'}{P}f'' + \frac{6}{P}f'^2 - \frac{4P'}{P^2}ff' + \frac{P''}{P^2}f^2 + \frac{4Q}{P^2}f' - \frac{2Q'}{P^2}f + \frac{2R}{P^2} = 0 \tag{9.0.1}$$

[44] where $Re\,\gamma$, $Re\,\delta > -1$, $Re\,\beta > -2\min\left(\frac{1}{n}, \frac{Re\,\gamma+1}{n-1}, \frac{Re\,\delta+1}{n-1}\right)$

with arbitrary polynomials $P(z), Q(z), R(z)$ of degree $3, 2, 1$ respectively. Cosgrove and Scoufis [24, 23], (A.3), show that this third order equation has a first integral, which is second order in f and quadratic in f'',

$$\begin{aligned} f''^2 \quad &+ \frac{4}{P^2}\Big((Pf'^2 + Qf' + R)f' - (P'f'^2 + Q'f' + R')f \\ &+ \frac{1}{2}(P''f' + Q'')f^2 - \frac{1}{6}P'''f^3 + c\Big) = 0; \end{aligned} \tag{9.0.2}$$

c is the integration constant. Equations of the general form

$$f''^2 = G(x, f, f')$$

are invariant under the map

$$x \mapsto \frac{a_1 z + a_2}{a_3 z + a_4} \quad \text{and} \quad f \mapsto \frac{a_5 f + a_6 z + a_7}{a_3 z + a_4}.$$

Using this map, the polynomial $P(z)$ can be normalized to

$$P(z) = z(z-1), \; z, \; \text{or } 1.$$

In this way, Cosgrove shows (9.0.2) is a master Painlevé equation, containing the 6 Painlevé equations. In each of the cases, the canonical equations are respectively:

- $g''^2 = -4g'^3 - 2g'(zg' - g) + A_1$ (**Painlevé II**)
- $g''^2 = -4g'^3 + 4(zg' - g)^2 + A_1 g' + A_2$ (**Painlevé IV**)
- $(zg'')^2 = (zg' - g)\Big(-4g'^2 + A_1(zg' - g) + A_2\Big) + A_3 g' + A_4$ (**Painlevé V**)
- $(z(z-1)g'')^2 = (zg' - g)\Big(4g'^2 - 4g'(zg' - g) + A_2\Big) + A_1 g'^2 + A_3 g' + A_4$ (**Painlevé VI**)

(9.0.3)

Painlevé II equation above can be solved by setting

$$\begin{aligned} g(z) &= \frac{1}{2}(u')^2 - \frac{1}{2}(u^2 + \frac{z}{2})^2 - (\alpha + \frac{\varepsilon_1}{2})u \\ g'(z) &= -\frac{\varepsilon_1}{2}u' - \frac{1}{2}(u^2 + \frac{z}{2}) \\ A_1 &= \frac{1}{4}(\alpha + (u^2 + \frac{z}{2})^2\varepsilon_1)^2, \quad (\varepsilon = \pm 1). \end{aligned}$$

Then $u(z)$ satisfies yet another version of the Painlevé II equation

$$u'' = 2u^3 + zu + \alpha. \qquad \textbf{(Painlevé II)}$$

Now, each of these Painlevé II,IV,V,VI equations can be transformed into the standard Painlevé equations, which are all differential equations of the form

$$f'' = F(z, f, f'), \text{rational in } f,\ f', \text{ analytic in } z,$$

whose general solution has no movable critical points. Painlevé showed that this requirement leads to 50 types of equations, six of which cannot be reduced to known equations.

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